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On-line Appendix for "Coalition Formation in Legislative Bargaining"

Abstract

In this appendix we present the proofs and sections omitted in "Coalition Formation in Legislative Bargaining."

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1 Proof of Proposition 4

As in the main text, we denote $x_j^i = x_j(\{i,j\}, i)$. We also denote $S_\tau(C)$ as the average surplus in coalition C when τ is the formateur, that is $S_\tau(C) = \frac{1}{2}\left[V(C) - (\sum_{j\in C} x_j^{\tau+1})\right]$. We have two cases to consider: when $d, e \geq 0$ and when $d, e < 0$.

1.1 Case 1: $d, e \geq 0$

We first prove that, in the limit equilibrium, if 2 forms a coalition with 1, then 1 forms a coalition with 3 and 3 with 2, thus we are in a clockwise equilibrium.

Lemma A.4.1. *If in the limit equilibrium as $\Delta \to 0$ party 2 forms a coalition $\{1,2\}$, then 1 forms $\{1,3\}$ and 3 forms $\{3,2\}$.*

Proof. Without loss of generality, we can assume that the same selections of coalitions as in the limit are made on the sequence of equilibria as $\Delta \to 0$ (otherwise we can select a subsequence with this property). Assume that in the limit equilibrium formateur 2 forms a coalition $\{2,1\}$ and, by contradiction, 1 forms a coalition $\{1,2\}$. Then, on the sequence, we have $x_3^2 = 0$ and $x_1^2 > 0$, $x_2^2 > 0$. By (6) in the paper, this implies that:

$$\begin{aligned} x_1^1 &= \xi x_1^2 + \phi(a - d - \xi x_1^2 - \xi x_2^2) < \xi x_1^2 + \phi(a - d - \xi x_1^2) \\ &\leq \xi x_1^2 + \phi(a + e - \xi x_1^2) = \widehat{x}_1^1 \end{aligned}$$

where $\xi = (\beta p) / [1 - \beta(1-p)]$ and $\phi = [1 - \beta(1-p)] / \left[1 - (\beta(1-p))^2\right]$. Note that $\widehat{x}_1^1$ is the payoff that 1 would obtain by forming a coalition with 3, a contradiction with the assumption that 1 chooses to form a coalition with 2. Assume that 2 forms with 1, and 1 does not form a coalition. Then 1 obtains ξx_1^2 that is strictly lower than what s/he would obtain forming a coalition with 2: $\xi x_1^2 + \phi(a - d - \xi x_1^2 - \xi x_2^2) = \xi x_1^2 + \phi(1-\xi)(a-d) > \xi x_1^2$. It must therefore be that 2 forms a coalition with 1, and 1 with 3. There are 2 possible cases to rule out: 3 is unable to form a coalition (Case 1); and that 3 forms a coalition with 1 (Case 2).

Case 1. In this case, 3 obtains ξx_3^1 that is strictly lower than what s/he would obtain forming a coalition with 1, $\xi x_3^1 + \phi(1-\xi)(a+e)$, a contradiction.

Case 2. In this case, as $\Delta \to 0$, we have:

$$\begin{aligned} x_1^1 &= \frac{a+e}{2} + \frac{x_1^2}{2},\ x_2^1 = 0,\ x_3^1 = \frac{a+e}{2} - \frac{x_1^2}{2} \\ x_1^2 &= \frac{a-d}{2} + \frac{x_1^3}{2},\ x_2^2 = \frac{a-d}{2} - \frac{x_1^3}{2},\ x_3^2 = 0 \\ x_1^3 &= \frac{a+e}{2} + \frac{x_1^1 - x_3^1}{2},\ x_2^3 = 0,\ x_3^3 = \frac{a+e}{2} - \frac{x_1^1 - x_3^1}{2}. \end{aligned}$$

We have that $x_1^1 - x_3^1 = x_1^2$, so: $x_1^3 = a + (2e-d)/3, x_2^3 = 0, x_3^3 = (e+d)/3$, and $x_1^2 = a + (e-2d)/3, x_2^2 = -(d+e)/3, x_3^2 = 0$. This implies $x_2^2 = -(d+e)/3 \leq 0$. If either $d > 0$ or $e > 0$, we have a contradiction. If $d = e = 0$, then $S_2(\{2,3\}) = a > S_2(\{1,2\}) = 0$, so we cannot have a limit equilibrium as $\Delta \to 0$ in which 2 forms $\{1,2\}$. ■

We now characterize the equilibria in the case in which 2 forms a coalition $\{2,3\}$.

Lemma A.4.2. *If, in the limit equilibrium as $\Delta \to 0$, 2 forms a coalition $\{2,3\}$, then 3 forms $\{1,3\}$ and 1 forms $\{1,2\}$.*

Proof. We proceed in 4 steps.

Step 1. Again, without loss of generality, we can assume that the same selections of coalitions as in the limit are made on the sequence of equilibria as $\Delta \to 0$ (otherwise we can select a subsequence with this property). We first show that if $\{2,3\}$ is formed when 2 is formateur, then $\{3,2\}$ can not form a coalition in equilibrium when 3 is formateur. Assume not. Assume first that 1 is unable to form a coalition. Consider a deviation in which 1 forms with 3. Since $x_1^2 = 0$, 1's payoff would be $\widehat{x}_1^1 = \phi(a + e - \xi(x_3^2)) \geq \phi(a + e - \xi a) > 0 = \xi x_1^2 = x_1^1$, where $\xi = (\beta p)/[1-\beta(1-p)]$ and $\phi = [1-\beta(1-p)]/\left[1-[\beta(1-p)]^2\right]$ and, in the first inequality, we use the fact that $x_2^2 + x_3^2 \leq a$, so a fortiori $x_3^2 \leq a$. This implies that 1 has a strictly profitable deviation, a contradiction. Assume then that 1 forms a coalition with 3. In this case, as $\Delta \to 0$:

$$\begin{aligned}
x_1^1 &= \frac{a+e}{2} - \frac{x_3^2}{2}, \, x_2^1 = 0, \, x_3^1 = \frac{a+e}{2} + \frac{x_3^2}{2} \\
x_1^2 &= 0, \, x_2^2 = \frac{a}{2} + \frac{x_2^3 - x_3^3}{2}, \, x_3^2 = \frac{a}{2} - \frac{x_2^3 - x_3^3}{2} \\
x_1^3 &= 0, \, x_2^3 = \frac{a}{2} - \frac{x_3^1}{2}, \, x_3^3 = \frac{a}{2} + \frac{x_3^1}{2}.
\end{aligned}$$

This implies that $x_3^2 = a + \frac{e}{3}$. If $e > 0$, we have a contradiction since we must have $x_3^2 \leq a$. If $e = 0$, then $x_2^2 = 0$, so $S_1(1,2) = a - d - x_2^2 = a - d > S_1(1,3) = 0$. In both cases, we cannot have a sequence of equilibria converging to an outcome in which 2 forms with 3, 3 with 2 and 1 with 3 as $\Delta \to 0$. Finally, assume that 1 forms a coalition with 2. In this case, as $\Delta \to 0$, we have:

$$\begin{aligned}
x_1^1 &= \frac{a-d}{2} - \frac{x_2^2}{2}, \, x_2^1 = \frac{a-d}{2} + \frac{x_2^2}{2}, \, x_3^1 = 0 \\
x_1^2 &= 0, \, x_2^2 = \frac{a}{2} + \frac{x_2^3 - x_3^3}{2}, \, x_3^2 = \frac{a}{2} - \frac{x_2^3 - x_3^3}{2} \\
x_1^3 &= 0, \, x_2^3 = \frac{a}{2} + \frac{x_2^1}{2}, \, x_3^3 = \frac{a}{2} - \frac{x_2^1}{2}.
\end{aligned}$$

In this case we have $x_2^2 = a - d/3$, $x_3^2 = d/3$, thus $S_1(\{1,2\}) = -\frac{1}{2}(2/3)d$, $S_1(\{1,3\}) = \frac{1}{2}[a + e - d/3] > S_1(\{1,2\})$. This implies that we cannot have a sequence of equilibria that converges as $\Delta \to 0$ to an equilibrium in which 2 forms with 3, 3 with 2 and 1 with 2.

Step 2. We now show that if $\{2,3\}$ is formed when 2 is formateur, then 3 must be able to form a coalition. Assume, by contradiction, that this is not the case and that either 1 is unable to form a coalition, or 1 forms a coalition with 3. In these case, by the same argument as in Lemma A.4.1, it must be that 3 finds it strictly optimal to form a coalition with, respectively, either 2 or 1. Assume then that 1 forms a coalition with 2. It follows that 3, by forming a coalition with 1, can obtain $\widehat{x}_3^3 = \phi\left[a + e - \xi\left(x_1^1 + x_3^1\right)\right] \geq \phi\left[a + e - \xi\left(a - d\right)\right] > 0 > \xi x_3^1 = x_3^3$, a contradiction.

Step 3. It must be that if 2 forms $\{2,3\}$, then 3 forms $\{1,3\}$. We now show that if $\{2,3\}$ is formed when 2 is formateur and $\{1,3\}$ is formed by 3, then 1 must be able to form a coalition. If this were not the case, then, as $\Delta \to 0$, we would have: $x_1^2 = 0$, $x_2^2 = a/2 - x_3^3/2$, $x_3^2 = a/2 + x_3^3/2$ and $x_1^3 = (a+e)/2 - x_3^2/2$, $x_2^3 = 0$, $x_3^3 = (a+e)/2 + x_3^2/2$. These equations imply: $x_3^3 = (3a + 2e)/3$, $x_1^3 = e/3$. Consider now the net surplus when 2 is the formateur: $S_2(\{1,2\}) = (a - d - e/3)/2 > 0$ and $S_2(\{2,3\}) = -(1/3)e \leq 0$. This implies that 2 would strictly prefer $\{1,2\}$ to $\{2,3\}$ in the limit as $\Delta \to 0$. This implies that we cannot have a sequence of equilibria with $\Delta > 0$ that converge to an equilibrium in which 2 forms with 3, 3 with 1 and 1 is unable or unwilling to form a coalition.

Step 4. We finally show that if 2 forms $\{2,3\}$ and 3 forms $\{3,1\}$, then 1 cannot form $\{1,3\}$. Assume not, then as $\Delta \to 0$, we would have:

$$\begin{aligned}
x_1^1 &= \frac{a+e}{2} - \frac{x_3^2}{2},\ x_2^1 = 0,\ x_3^1 = \frac{a+e}{2} + \frac{x_3^2}{2} \\
x_1^2 &= 0,\ x_2^2 = \frac{a}{2} - \frac{x_3^3}{2},\ x_3^2 = \frac{a}{2} + \frac{x_3^3}{2} \\
x_1^3 &= \frac{a+e}{2} + \frac{x_1^1 - x_3^1}{2},\ x_2^3 = 0,\ x_3^3 = \frac{a+e}{2} - \frac{x_1^1 - x_3^1}{2},
\end{aligned}$$

The first and last equation in the first line imply $x_1^1 - x_3^1 = -x_3^2$. The last equation in the third line and the third in the second line imply: $x_3^2 = a + e/3$. If $e > 0$, we have a contradiction since $x_3^2 \leq a$. If $e = 0$ we have $S_2(\{1,2\}) = a - d > S_2(\{2,3\}) = 0$, again this implies we cannot have a limit equilibrium in which 2 forms $\{2,3\}$, a contradiction.

Steps 1-4 imply that if $\{2,3\}$ is formed when 2 is formateur, then 3 must form $\{1,3\}$ and 1 must form $\{1,2\}$, so we have a clockwise equilibrium. ■

We conclude the proof showing that if 2 is unable to form a coalition, then no other party will choose to form a coalition with 2. This implies that the equilibrium must be strongly efficient.

Lemma A.4.3. *If, in the limit equilibrium as $\Delta \to 0$, 2 is unable or unwilling to form a coalition, then no other party chooses to form a coalition with 2.*

Proof. Note first that a coalition must form in equilibrium. If this were not the case, then the reservation utilities would all be equal to 0, leading to a contradiction. Assume that 3 forms with

2, but 2 is unwilling or unable to form a coalition with any party. Formateur 2's expected payoff when formateur is $[\beta p/ (1-\beta(1-p))]\, x_2^3$. Assume instead that 2 instead forms a government with 3. Party 2's payoff would be: $\widehat{x}_2^2 = \xi x_2^3 + \phi(a - \xi\left(x_2^3 + x_3^3\right)) = \xi x_2^3 + \phi(1-\xi)a > \xi x_2^3$, where $\xi = (\beta p)/[1-\beta(1-p)]$ and $\phi = [1-\beta(1-p)]/\left[1-(\beta(1-p))^2\right]$ and in the equality we use the fact that $x_2^3 + x_3^3 = a$. So we have a contradiction. Assume then that 1 forms a government with 2. There are two other possible sub-cases. First, the case in which 3 is unable or unwilling to form a coalition. In this case 2's payoff is ξx_2^1 which, by a similar argument as above, is strictly lower than the payoff that 2 can obtain by forming a government with 1, a contradiction. Second, the case in which 3 forms a government with 1. Since offering to 2 instead of 3 must be optimal for 1, we must have: $x_1^1 = \xi x_1^3 + \phi(a - d - \xi\left(x_1^3 + x_2^3\right)) \geq \xi x_1^3 + \phi(a + e - \xi\left(x_1^3 + x_3^3\right)) = \widehat{x}_1^1$, where $\widehat{x}_1^1$ is the payoff 1 would obtain by forming a government with 3. So $a - d - \xi\left(x_1^3 + x_2^3\right) \geq a + e - \xi\left(x_1^3 + x_3^3\right) = (1-\xi)(a+e) > 0$. But then 2 can form a coalition with 1 and obtain $\xi x_2^3 + \phi(a - d - \xi\left(x_1^3 + x_2^3\right))$, which is strictly more than the payoff in case of failure of forming a coalition, ξx_2^3, a contradiction. ■

1.2 Case 2: $d, e < 0$

As for Case 1, we proceed in three steps. First, we show that the clockwise equilibrium is the only possible limit equilibrium when 3 forms a government with 2; we then show that the only type of limit equilibrium in which 3 forms a government with 1 is a counter-clockwise equilibrium. Finally, we show that if 3 is unable or unwilling to form a government, then no other party forms a government with 3, thus we have a strongly efficient equilibrium.

Step 1. As above, we can assume without loss of generality that the same selections of coalitions as in the limit are made on the sequence of equilibria converging to the limit equilibrium as $\Delta \to 0$ (otherwise we can select a subsequence with this property). Assume that 3 forms a coalition with 2. We show that it can not be that 2 forms a government with 3. In this case we have 2 obtains a payoff $x_2^2 = \xi x_2^3 + \phi(a - \xi\left(x_2^3 + x_3^3\right)) < \xi x_2^3 + \phi(a - d - \xi x_2^3) = \widehat{x}_2^2$, where $\widehat{x}_2^2$ is the payoff that would be obtained by 2 deviating and forming $\{1, 2\}$, a contradiction. By a similar argument as the arguments used in the previous subsection, it must be that 2 forms a coalition with some other party, so if 3 forms with 2, then 2 must form a coalition with 1. We now show that if 3 forms with 2 and 2 forms with 1, then 1 can must form a coalition with 3. Again, it must be that if 3 forms with 2 and 2 forms with 1, then 1 must be able to form a coalition (since by failing to form a coalition, 1 would receive less than by forming a coalition with 2). So we only need to show that it cannot be that 1 forms $\{1, 2\}$. Assume by contradiction that this is not the case, then in the limit as $\Delta \to 0$:

$$\begin{aligned} x_1^1 &= \frac{a-d}{2} + \frac{x_1^2 - x_2^2}{2},\ x_2^1 = \frac{a-d}{2} - \frac{x_1^2 - x_2^2}{2},\ x_3^1 = 0 \\ x_1^2 &= \frac{a-d}{2} - \frac{x_2^3}{2},\ x_2^2 = \frac{a-d}{2} + \frac{x_2^3}{2},\ x_3^2 = 0 \\ x_1^3 &= 0,\ x_2^3 = \frac{a}{2} + \frac{x_2^1}{2},\ x_3^3 = \frac{a}{2} - \frac{x_2^1}{2}. \end{aligned}$$

These equations imply that: $x_1^1 = -d/3$, $x_2^1 = (3a - 2d)/3$, $x_3^1 = 0$, $x_1^2 = -d/3$, $x_2^2 = a - (2/3)d$, $x_3^2 = 0$, and $x_1^3 = 0$, $x_2^3 = a - d/3$, $x_3^3 = d/3$. Note that $x_2^3 = a - d/3 > a$ and $x_3^3 = d/3 < 0$. But then $x_2^3 > a$, which is impossible. We conclude we cannot have a limit equilibrium in which 3 forms with 2, 2 forms with 1, and 1 forms with 2. It must be that if 3 forms a coalition with 2, then 2 forms a coalition with 1 and 1 forms a coalition with 3.

Step 2. Assume now that 3 forms a coalition with 1. Assume first that 1 forms with 3. In this case we have: $x_3^3 = \xi x_3^1 + \phi(a + e - \xi\left(x_1^1 + x_3^1\right)) < \xi x_3^1 + \phi(a - \xi x_3^1) = \widehat{x}_3^3$, where $\widehat{x}_3^3$ is the payoff that would be obtained by 3 deviating and forming $\{2,3\}$, a contradiction. We now show that 1 must form a government with 2. Assume by contradiction, first, that neither 1 nor 2 forms a coalition, then 2 would obtain $\xi x_2^3 = 0$, but s/he would be able to obtain $\phi(a - \xi x_3^3) > \phi(a - \xi\left(a + e\right)) > \phi(1-\xi)a > 0$ by forming a coalition with 3, a contradiction. Assume now that 1 is unable to form a coalition and 2 forms a coalition with 3. As $\Delta \to 0$, we must have: $x_1^2 = 0$, $x_2^2 = \frac{a}{2} - \frac{x_3^3}{2}$, $x_3^2 = \frac{a}{2} + \frac{x_3^3}{2}$ and $x_1^3 = (a+e)/2 - x_3^2/2$, $x_2^3 = 0$, $x_3^3 = (a+e)/2 + x_3^2/2$. This gives us $x_3^3 = a + (2/3)e > a + e$, a contradiction. Finally, assume that 1 is unable to form a coalition and 2 forms a coalition with 1. In this case, as $\Delta \to 0$, we have: $x_1^2 = \frac{a-d}{2} + \frac{x_1^3}{2}$, $x_2^2 = \frac{a-d}{2} - \frac{x_1^3}{2}$, $x_3^2 = 0$ and $x_1^3 = (a+e)/2 + x_1^2/2$, $x_2^3 = 0$, $x_3^3 = (a+e)/2 - x_1^2/2$. This gives us $x_1^2 = (3a - 2d + e)/3$, $x_1^3 = (3a + 2e - d)/3$ and $x_3^3 = (d+e)/3 < 0$, a contradiction. We conclude that if 3 forms a coalition with 1, then 1 forms a coalition with 2. Assume now that 2 is unable to form a coalition. In this case, as $\Delta \to 0$, we have: $x_1^1 = (a-d)/2 + x_1^3/2$, $x_2^1 = (a-d)/2 - x_1^3/2$, $x_3^1 = 0$ and $x_1^3 = (a+e)/2 + x_1^1/2$, $x_2^3 = 0$, $x_3^3 = (a+e)/2 - x_1^1/2$. So $x_1^1 = a + (e - 2d)/3$ and $x_3^3 = (a+e)/2 - a/2 - (e-2d)/6 = (2e+2d)/6 < 0$, a contradiction. Finally assume that 3 forms a coalition with 1, 1 forms a coalition with 2 and 2 forms a coalition with 1. As $\Delta \to 0$, we have:

$$\begin{aligned} x_1^1 &= \frac{a-d}{2} + \frac{x_1^2 - x_2^2}{2},\ x_2^1 = \frac{a-d}{2} - \frac{x_1^2 - x_2^2}{2},\ x_3^1 = 0 \\ x_1^2 &= \frac{a-d}{2} + \frac{x_1^3}{2},\ x_2^2 = \frac{a-d}{2} - \frac{x_1^3}{2},\ x_3^2 = 0. \\ x_1^3 &= \frac{a+e}{2} + \frac{x_1^1}{2},\ x_2^3 = 0,\ x_3^3 = \frac{a+e}{2} - \frac{x_1^1}{2} \end{aligned}$$

so $x_1^1 = a + \frac{e-2d}{3}$ and $x_3^3 = \frac{1}{3}(e+d) < 0$, a contradiction. We conclude that if in the limit equilibrium 3 forms $\{1,3\}$, then the equilibrium is a counter-clockwise.

Step 3. We now show that if in the limit equilibrium 3 is unable or unwilling to form a government, then no other party forms a government with 3, thus a coalition $\{1,2\}$ is formed. We have 2 cases to rule out:

Case 1. Assume that in the limit equilibrium 3 is unable to form a government and 1 forms $\{1,3\}$. Then, on the sequence of equilibria as $\Delta \to 0$, $x_3^3 = \xi x_3^1$, but by forming a coalition with 1, 3 obtains $\xi x_3^1 + \phi(1-\xi)(a+e) > \xi x_3^1$, a contradiction.

Case 2. Assume that 3 is unable to form a government and 2 forms $\{2,3\}$. If 1 is unable to form a government too, then we have $x_3^3 = \xi^2 x_3^2$, but by forming a coalition with 2, 3 obtains $\xi^2 x_3^2 + \phi\left(a - \xi^2 x_3^2 - \xi^2 x_2^2\right) = \xi^2 x_3^2 + \phi\left(1-\xi^2\right) a > \xi^2 x_3^2$, a contradiction. If 1 forms a government with 2, as $\Delta \to 0$, we have: $x_1^1 = (a-d)/2 - x_2^2/2$, $x_2^1 = (a-d)/2 + x_2^2/2$, $x_3^1 = 0$, and $x_1^2 = 0$, $x_2^2 = a/2 + x_2^1/2$, $x_3^2 = a/2 - x_2^1/2$. Solving the system, we obtain: $x_1^1 = -d/3$, $x_2^1 = (3a-2d)/3$, $x_3^1 = 0$ and $x_1^2 = 0$, $x_2^2 = (3a-d)/3$, $x_3^2 = d/3$. But then $x_3^2 = d/3 < 0$, a contradiction. ■

From the Steps presented above, we conclude that either 3 is able to form a coalition, in which case the equilibrium is clockwise or counterclockwise; or 3 is unable to form a coalition, in which case the only coalition that can be formed in equilibrium is $\{1,2\}$, so we have a strongly efficient equilibrium. ■

2 Proof of Proposition 5

We complete the proof of Proposition 5 by proving Lemma A.5.2.

Lemma A.5.2. *A counter-clockwise equilibrium exists as $\Delta \to 0$ only if $d \leq \frac{3}{7}a - \frac{5}{7}e$ when $d,e > 0$, and if $d \geq -\frac{3}{5}a - \frac{1}{5}e$ and $d \leq 3a + 7e$ when $d,e \leq 0$; and it exists and is the limit of equilibria as $\Delta \to 0$ if these conditions are strict inequalities.*

Proof. As in the previous lemma, we proceed in three steps.

Step 1. We first construct the value functions associated to a counterclockwise equilibrium. From (6) in the paper, if counterclockwise equilibrium exists, we must have:

$$\begin{aligned} x_1^1 &= \xi x_1^2 + \phi\left(a - d - \xi x_1^2 - \xi x_2^2\right), \; x_2^1 = \xi x_2^2 + \phi\left(a - d - \xi x_1^2 - \xi x_2^2\right), \; x_3^1 = 0, \\ x_1^2 &= 0, \; x_2^2 = \xi x_2^3 + \phi\left(a - \xi x_2^3 - \xi x_3^3\right), \; x_3^2 = \xi x_3^3 + \phi\left(a - \xi x_2^3 - \xi x_3^3\right), \\ x_1^3 &= \xi x_1^1 + \phi\left(a + e - \xi x_1^1 - \xi x_3^1\right), \; x_2^3 = 0, \; x_3^3 = \xi x_3^1 + \phi\left(a + e - \xi x_1^1 - \xi x_3^1\right), \end{aligned} \tag{1}$$

where $\xi = \beta p / [1-\beta(1-p)]$ and $\phi = [1-\beta(1-p)] / \left[1-(\beta(1-p))^2\right]$. The equations in (1) give us a system of 9 equations in 9 unknowns, admitting a unique solution that, as $\Delta \to 0$, converges to:

$$\begin{aligned} x_1^1 &= \frac{3a-4d+e}{9},\ x_2^1 = \frac{6a-5d-e}{9},\ x_3^1 = 0 \\ x_1^2 &= 0,\ x_2^2 = \frac{3a-d-2e}{9},\ x_3^2 = \frac{6a+d+2e}{9} \\ x_1^3 &= \frac{6a-2d+5e}{9},\ x_2^3 = 0,\ x_3^3 = \frac{3a+2d+4e}{9}, \end{aligned} \tag{2}$$

A necessary condition for the counterclockwise strategies to be an equilibrium as $\Delta \to 0$ is that no party has a profitable deviation in the limit given (2). A sufficient condition for the counterclockwise strategies to be the limit of equilibria as $\Delta \to 0$ is that they are a strict equilibrium in the limit.

Step 2. We now characterize the conditions under which no party has a profitable deviation in the limit as $\Delta \to 0$ when $d, e \geq 0$. Consider first the case in which 1 is the formateur. We have: $S_1(\{1,2\}) = \left(a-d-x_1^2-x_2^2\right)/2 = (6a+2e-8d)/18 > 0$ and $S_1(\{1,3\}) = \left(a+e-x_1^2-x_3^2\right)/2 = (3a+7e-d)/18$. It follows that $S_1(\{1,2\}) \geq S_1(\{1,3\})$ if and only if $d \leq (3/7)\,a - (5/7)\,e$, and the first inequality is strict if the second is strict. Moreover, $S_1(\{1,2\}) \geq 0$ if $d \leq (3/4)\,a + e/4$, which is implied by $d \leq (3/7)\,a - (5/7)\,e$ for $d \geq 0$. Consider now formateur 2. We have: $S_2(\{1,2\}) = \left(a-d-x_1^3\right)/2 = (3a-7d-5e)/18 \geq 0$ and $S_2(\{2,3\}) = \left(a-x_3^3\right)/2 = (6a-2d-4e)/18$, so $S_2(\{2,3\}) > S_2(\{1,2\})$ is always true. Finally, consider now formateur 3's decision: $S_3(\{1,3\}) = \left(a+e-x_1^1-x_3^1\right)/2 = (6a+8e+4d)/18 > 0$ and $S_3(\{2,3\}) = \left(a-x_2^1-x_3^1\right)/2 = (3a+5d+e)/18$, so $S_3(\{1,3\}) \geq S_3(\{2,3\})$ is true if $d \leq 3a+7e$, which holds whenever $d \leq (3/7)a - (5/7)e$. We conclude that a counter-clockwise equilibrium exists only if $d \leq (3/7)a - (5/7)e$ and it exists and is the limit of equilibria as $\Delta \to 0$ if $d < (3/7)a - (5/7)e$.

Step 3. Assume now $d, e < 0$ and consider first the case in which 2 is the formateur. We have: $S_2(\{1,2\}) = \left(a-d-x_1^3\right)/2 = (3a-7d-5e)/18 > 0$ and $S_2(\{2,3\}) = \left(a-x_3^3\right)/2 = (6a-2d-4e)/18$: so $S_2(\{2,3\}) \geq S_2(\{1,2\})$ if $d \geq -(3/5)a - (1/5)e$, with the first inequality strict if the second is strict. Consider now formateur 1. We have: $S_1(\{1,2\}) = \left(a-d-x_2^2\right)/2 = (6a-8d+2e)/18 > 0$ and $S_1(\{1,3\}) = \left(a+e-x_3^2\right)/2 = (3a+7e-d)/18$. It follows that $S_1(\{1,2\}) \geq S_1(\{1,3\})$ if and only if $3a-7d-5e \geq 0$, where again the first inequality is strict if the second is strict. Finally, consider now Formateur 3's decision: $S_3(\{1,3\}) = \left(a+e-x_1^1\right)/2 = (6a+4d+8e)/18$ and $S_3(\{2,3\}) = \left(a-x_2^1\right)/2 = (3a+5d+e)/18$. It follows that $S_3(\{1,3\}) \geq S_3(\{2,3\})$ if $3a+7e-d \geq 0$, where the first inequality is strict if the second is strict. Note that $3a-7d+5e > 3a+7e-d$ for $d < 0$, thus the condition on formateur 3 implies the condition on formateur 1. We conclude that a counterclockwise equilibrium exists only if $d \geq -(3/5)a - (1/5)e$ and $d \leq 3a + 7e$, and it exists and is the limit of equilibria as $\Delta \to 0$ if these two inequalities are strict. ■

3 The mixed strategy equilibrium presented in Section 4

We construct here a mixed strategy equilibrium for the region defined by $d < -\frac{3}{2}a - 2e$ and $d > -(a+e)$ with $d, e < 0$ in which, as discussed in Section 4 of the paper, a pure strategy equilibrium does not exist. For this region we construct an equilibrium in which 1 chooses 2 with probability α, and 3 with probability $1-\alpha$; 2 chooses 1 with probability 1; and 3 chooses 2 with probability 1 (see lower right panel in Figure 2). We must have: $x_3^1 = (a+e)/2 - x_1^2/2$ and $x_2^1 = (a-d)/2 + \left(x_2^2 - x_1^2\right)/2$. The indifference condition for 1 is: $(a+e)/2 + x_1^2/2 = (a-d)/2 - \left(x_2^2 - x_1^2\right)/2$, implying $x_2^2 = -(e+d)$, $x_1^2 = a - d - x_2^2 = a + e$, $x_3^2 = 0$.

We also must have $x_3^1 = 0$, $x_1^1 = (a+e)/2 + x_1^2/2 = a + e$ and $x_2^1 = a - d - x_1^1 = -(e+d)$. Moreover, $x_1^3 = 0$, $x_2^3 = a/2 + \left(\alpha x_2^1 - (1-\alpha)x_3^1\right)/2 = a/2 - (\alpha/2)\,x_2^1 = (a/2) - (\alpha/2)\,(e+d)$. And we have: $x_2^2 = (a-d)/2 + \left(x_2^3 - x_1^3\right)/2 = (a-d)/2 + x_2^3/2$. Thus we have $(a-d)/2 + x_2^3/2 = -(d+e)$, implying $x_2^3 = -(d+2e+a)$. Substituting in the formula above we have:

$$\alpha = \frac{3a + 2d + 4e}{e + d}$$

Note that since $d + e < 0$, $\alpha \geq 0$ for $d \leq -\frac{3}{2}a - 2e$. Moreover $\alpha \leq 1$ for $3a + 2d + 4e \leq e + d$, that is if $d \geq -3(a+e)$. It can be verified tat these conditions are always satisfied in the region of interest.

Obviously the strategy is optimal for formateur 1 by construction. For formateur 2, it is optimal if $S_2(1,2) \geq S_2(2,3)$. We have: $S_2(1,2) = (a - d - x_1^3 - x_2^3)/2 = 2(a+e) \geq 0$ and $S_2(2,3) = (a - x_2^3 - x_3^3)/2 = 0$. The condition is therefore verified.

For formateur 3 we need $S_3(2,3) \geq S_3(1,3)$. We have $S_3(1,3) = (a + e - x_1^1 - Ex_3^1)/2 = 0$ and:

$$\begin{aligned} S_3(2,3) &= \frac{1}{2}(a - Ex_2^1 - Ex_3^1) = a - \alpha(e+d) \\ &= \frac{1}{2}\left[a + \frac{3a + 2d + 4e}{e + d}(e+d)\right] = 2a + d + 2e \end{aligned}$$

Thus the condition is verified if $d \geq -2(a+e)$. Note that in the relevant region we need to have: $d \geq -\frac{3}{2}a - 2e > -2(a+e)$, thus this condition is always satisfied. ■

4 Proof of Proposition 6

The fact that a clockwise equilibrium exists as a limit equilibrium follows from the argument in Section 6.1. We now prove that when the core is empty, no other pure strategy equilibrium exists as $\Delta \to 0$. Lemma A.6.2 deals with the case with $d, e \geq 0$, Lemma A.6.3 with the case with $d, e < 0$.

Lemma A.6.2. *Assume $d, e \geq 0$, TIOLI offers and an empty core. In a limit equilibrium in pure strategies, party 1 forms a coalition $\{1,3\}$, party 2 forms $\{1,2\}$ and party 3 forms $\{3,2\}$.*

Proof. We proceed in five steps.

Step 1. We first show that if $a > |e|+|d|$, then it cannot be that any party fails to form a coalition and is then excluded by the first coalition that is formed in equilibrium after its turn. If 2 fails to form a coalition and is excluded by the first coalition that is formed in equilibrium after its turn, then we must have $\beta x_1^3 \geq a - d$ and $\beta x_3^3 \geq a$ (else 2 would be be able to make a profitable offer that would be accepted by some coalition). It follows that $\beta \left(x_1^3 + x_3^3\right) \geq 2a - d$. We however must have that: $\beta (a + e) \geq \beta \left(x_1^3 + x_3^3\right) \geq 2a - d$, implying, as $\Delta \to 0$, $a \leq e + d$, a contradiction. If 1 fails to form a coalition and it is subsequently excluded, we must have $\beta x_2^2 \geq a - d$ and $\beta x_3^2 \geq a + e$, so: $\beta a \geq \beta \left(x_2^2 + x_3^2\right) \geq a + e$, which is impossible for any $\beta < 1$. Finally, if 3 fails to form a coalition and it is subsequently excluded, we must have $\beta x_1^1 \geq a + e$ and $\beta x_2^1 \geq a$, so: $\beta (a - d) \geq \beta \left(x_1^1 + x_2^1\right) > a + e$, again impossible for $\beta < 1$.

Step 2. We now show that party 1 can not fail to form a coalition. By Step 1 it can not be that 1 fails and 2 forms a coalition with 3. It also can not be that 1 fails to form a coalition and 2 forms a coalition with to 1: since in this case $x_3^2 = 0$, so 1 would be able to form a coalition with 3 and obtain $a + e \geq a - d > \beta x_1^2 = x_1^1$, a contradiction. And it can not be that all parties fail, since in this case $x_3^2 = 0$, so $x_1^1 \geq a + e > 0$, a contradiction. So it must be that 2 also fails and, by Step 1, 3 forms a coalition with 1. But this is impossible by Step 1 again, since it would imply that 2 fails and is not included in the subsequent equilibrium coalition.

Step 3. We now show that there is only one limit equilibrium in pure strategies in which 1 forms a coalition with 3. In this equilibrium, 1 forms a coalition with 3, 3 forms a coalition with 2 and 2 with 1 (i.e. it is a clockwise equilibrium). In this equilibrium, moreover, the formateurs extracts all the surplus. In the following sub-steps, assume that 1 forms a coalition with 3.

Step 3.1. Assume first that 3 fails to form a coalition. In this case, 3 could form a coalition with 1 and get $x_3^1 + \frac{1}{2}(1 - \beta)(a + e) > x_3^1$, a contradiction.

Step 3.2. Assume by contradiction that 3 forms a coalition with 1. By Step 1, it can not be that 2 is unable to form a coalition: so, either 2 forms a coalition with 3 or with 1. Consider the case in which 2 forms with 3. As $\beta \to 1$, we must have: $x_1^1 = a + e - x_3^2$, $x_2^1 = 0$, $x_3^1 = x_3^2$, $x_1^2 = 0$, $x_2^2 = a - x_3^3$, $x_3^2 = x_3^3$, and $x_1^3 = x_1^1$, $x_2^3 = 0$, $x_3^3 = a + e - x_1^1$. This implies: $x_1^1 + x_3^3 = a + e$. Moreover, we must have $x_3^3 \geq a$, else 3 would deviate and form a coalition with 2; and $x_1^1 \geq a - d - x_2^2 \Leftrightarrow x_1^1 \geq (a + e - d)/2$, else 1 would prefer to form a coalition with 2. We therefore must have: $x_1^1 + x_3^3 \geq (3a + e - d)/2$. These inequalities are feasible only if $(3a + e - d)/2 \leq a + e$, that is $a \leq e + d$, a contradiction. Assume, again by contradiction, that 1 offers to 3, 3 to 1 and 2 to 1. As $\beta \to 1$, we have: $x_1^1 = a + e - x_3^2$, $x_2^1 = 0$, $x_3^1 = x_3^2$, $x_1^2 = x_1^3$, $x_2^2 = a - d - x_1^3$, $x_3^2 = 0$, and $x_1^3 = x_1^1$, $x_2^3 = 0$, $x_3^3 = a + e - x_1^1$. This implies: $x_1^1 + x_3^3 = a + e$ and $x_1^1 \geq a + e$. Moreover, we must have: $x_3^3 \geq a$, else 3 would deviate and form coalition with 2. This is feasible only if $2a + e \leq a + e$, i.e. $a \leq 0$, impossible.

Step 3.3. It therefore must be that if 1 forms a coalition with 3, then 3 forms a coalition with 2. It cannot be that 2 is unable to form a coalition: in this case $x_2^2 = \beta x_2^3$, but 2 can form a coalition with 3 and obtain $a - \beta x_3^3 > \beta \left(a - x_3^3\right) = \beta x_2^3$. Assume first that 2 forms a coalition with 3. In the limit as $\beta \to 1$, we have:

$$\begin{aligned} x_1^1 &= a + e - x_3^2,\ x_2^1 = 0,\ x_3^1 = x_3^2 \\ x_1^2 &= 0,\ x_2^2 = a - x_3^3,\ x_3^2 = x_3^3 \\ x_1^3 &= 0,\ x_2^3 = x_2^1,\ x_3^3 = a - x_2^1 \end{aligned}$$

We have: $x_1^1 = a + e - a + x_2^1 = e$, $x_2^2 = 0$, $x_3^3 = a$ and $x_1^3 = 0$. But then 2 could offer offer 0 to 1 and forms $\{1, 2\}$, making $a - d > 0$, a contradiction. We conclude that the only possibility is that 1 forms with 3, 3 with 2 and 2 with 1. In this case, as $\beta \to 1$, we have: $x_1^1 = a + e - x_3^2$, $x_2^1 = 0$, $x_3^1 = x_3^2$, $x_1^2 = x_1^3$, $x_2^2 = a - d - x_1^3$, $x_3^2 = 0$, and $x_1^3 = 0$, $x_2^3 = x_2^1$, $x_3^3 = a - x_2^1$. This system has a unique solution with value $x_1^1 = a + e$, $x_2^2 = a - d$, $x_3^3 = a$. It is easy to verify that this is an equilibrium for any $d, e \geq 0$.

Step 4. We now show that, if $a > |e| + |d|$, then there is no equilibrium in which 1 forms a coalition with 2. Assume by contradiction that 1 forms a coalition with 2. Clearly we cannot have that 2 forms with 1: if this were the case, then $x_3^2 = 0$, and 1 would like to form with 3 since $a + e > a - d - x_2^1$. It also can not be that 2 fails to form a coalition. To see this note that, by Step 1, it can not be that 3 offers to 1; or that both 2 and 3 fail to form a coalition (since in this case 3 would fail and excluded by 1). So it must be that 3 offers to 2. But then we must have $x_3^3 + x_2^3 \leq a$ so, by offering to 3, 2 obtains $\widehat{x}_2^2 = a - \beta x_3^3 > \beta x_2^3 = x_2^2$, a contradiction. So we must have that 1 offers to 2 and 2 to 3. By Step 1 is impossible that 1 offers to 2, 2 offers to 3 and 3 fails to form a coalition. Assume that 3 forms a coalition with 2. In this case, as $\beta \to 1$, we have:

$$\begin{aligned} x_1^1 &= a - d - x_2^2,\ x_2^1 = x_2^2,\ x_3^1 = 0 \\ x_1^2 &= 0,\ x_2^2 = a - x_3^3,\ x_3^2 = x_3^3 \\ x_1^3 &= 0,\ x_2^3 = x_2^1,\ x_3^3 = a - x_2^1 \end{aligned}$$

We must have that $x_2^2 + x_3^3 = a$. Moreover $x_2^2 \geq a - d$, else 2 may deviate and form a coalition with 1; and $x_3^3 \geq a + e - x_1^1$, else 3 could form a coalition with 1. This last inequality implies that $x_3^3 \geq a + e - a + d + x_2^2 = e + d + x_2^2 \geq e + d + a - d = a + e$. It follows that we have $a = x_2^2 + x_3^2 \geq 2a + e - d > a$, a contradiction.

Step 5. The only remaining possibility is that the limit equilibrium is a counterclockwise equilibrium. Consider a sequence $\beta_n > 0$ with $\beta_n \to 1$ and associated equilibria. Without loss of generality, along the sequence of equilibria 1 selects 2, 2 selects 3 and 3 selects 1 (if this is not the case, we can select a subsequence with this property). The equilibrium conditions on this sequence are:

$$\begin{aligned} x_1^1(n) &= a - d - \beta_n x_2^2(n),\ x_2^1(n) = \beta_n x_2^2(n),\ x_3^1(n) = 0 \\ x_1^2(n) &= 0,\ x_2^2 = a - \beta_n x_3^3(n),\ x_3^2 = \beta_n x_3^3(n) \\ x_1^3(n) &= \beta_n x_1^1(n),\ x_2^3(n) = 0,\ x_3^3(n) = a + e - \beta_n x_1^1(n) \end{aligned} \tag{3}$$

We therefore have:

$$x_1^1(n) = \frac{(1 - \beta_n + \beta_n^2)a - d + \beta_n^2 e}{1 + \beta_n^3},\ x_3^3(n) = \frac{\beta_n d + (1 - \beta_n(1 - \beta_n))a + e}{1 + \beta_n^3},$$

$$x_3^2(n) = \frac{\beta_n^2 d + (\beta_n - \beta_n^2(1 - \beta_n))a + \beta_n e}{1 + \beta_n^3}$$

Consider a deviation for party 1. If 1 deviates and forms a coalition with 3, s/he obtains:

$$\widehat{x}_1^1(n) = a + e - \beta_n x_3^2(n) = \frac{(1 - \beta_n^2 + 2\beta_n^3 - \beta_n^4)a + \left(1 - \beta_n^2 + \beta_n^3\right)e - \beta_n^3 d}{1 + \beta_n^3}$$

The deviation is profitable if for a sufficiently high $\beta_n < 1$:

$$\widehat{x}_1^1(n) > x_1^1(n) = \left[(1 - \beta_n + \beta_n^2)a - d + \beta_n^2 e\right]/(1 + \beta_n^3).$$

Note that we can write:

$$\widehat{x}_1^1(n) = x_1^1(n) + \frac{\left(1 - \beta_n^3\right)}{1 + \beta_n^3}\left[\beta_n \frac{1 - 2\beta_n + 2\beta_n^2 - \beta_n^3}{1 - \beta_n^3}a + \frac{1 - 2\beta_n^2 + \beta_n^3}{1 - \beta_n^3}e + d\right]$$

It follows that, as $\beta_n \to 1$, 1 has a strictly optimal deviation if and only if the sign of limit of the square parenthesis is positive. Applying l'Hospital rule, we have:

$$\beta_n \frac{1 - 2\beta_n + 2\beta_n^2 - \beta_n^3}{1 - \beta_n^3} a + \frac{1 - 2\beta_n^2 + \beta_n^3}{1 - \beta_n^3} e + d \to \frac{1}{3}(a + e) + d$$

When $d \geq 0$, $e \geq 0$ this term is always strictly positive, so 1 has a strictly positive deviation along any sequence of $\beta_n \to 1$. ■

We now turn to the case with $d, e < 0$.

Lemma A.6.3. *Assume $d, e < 0$, TIOLI offers and an empty core. In a limit equilibrium in pure strategies, party 1 forms a coalition $\{1, 3\}$, party 2 forms $\{1, 2\}$ and party 3 forms $\{3, 2\}$.*

Proof. As above, we proceed in 5 steps.

Step 1. We first show that if $a > |e| + |d|$, it cannot be that any party fails to form a coalition and is then excluded by the first coalition that is formed in equilibrium after his turn. If $d, e < 0$, then this condition can be written as $a > -(e + d)$. If 2 fails to form a coalition and it is subsequently excluded, we must have $\beta x_1^3 \geq a - d$ and $\beta x_3^3 \geq a$, so we must have $\beta (a + e) \geq \beta \left(x_1^3 + x_3^3\right) \geq 2a - d$, implying that, as $\beta \to 1$, $a \leq d + e < 0$, a contradiction. If 1 fails to form a coalition and it is subsequently excluded, we must have $\beta x_2^2 \geq a - d$ and $\beta x_3^2 \geq a + e$, so: $\beta \left(x_2^2 + x_3^2\right) \geq 2a - d + e > a - d$, impossible. Finally, if 3 fails to form a coalition and it is subsequently excluded, we must have $\beta x_1^1 \geq a + e$ and $\beta x_2^1 \geq a$, so: $\beta \left(x_1^1 + x_2^1\right) \geq 2a + e$. We must therefore have $\beta (a - d) \geq \beta \left(x_1^1 + x_2^1\right) \geq 2a + e$, implying as $\beta \to 1$, $a \leq |e| + |d|$, a contradiction.

Step 2. We now show that party 1 can not fail to form a coalition. By Step 1 it can not be that 1 fails and 2 offers to 3. It also can not be that all parties fail, since in this case $x_2^2 = 0$, so $x_1^1 \geq a - d > 0$, a contradiction. Assume that 1 fails and 2 forms a coalition with to 1. Then 1 would have a strict deviation by forming a coalition with 2. So it must be that 2 also fails and, by Step 1, 3 forms a coalition with 1. But this is impossible by Step 1 again since it would imply that 2 fails and is not included in the subsequent equilibrium coalition.

Step 3. We now show that there is only one limit equilibrium in pure strategies in which 1 forms a coalition with 3. In this equilibrium, 1 forms a coalition with 3, 3 forms with 2 and 2 with 1 (i.e. it is a clockwise equilibrium). In this equilibrium, moreover, the formateurs extracts all the surplus. In the following sub-steps, assume that 1 forms a coalition with 3.

Step 3.1. Assume first, by contradiction, that 3 fails to form a coalition. Then we have $x_3^3 = \beta x_3^1$; but, by making a TIOLI to 1, 3 can obtain $\widehat{x}_3^3 = a + e - \beta x_1^1 > \beta x_3^1$, a contradiction. Assume therefore that 3 forms a coalition with 1. By Step 1, it can not be that 2 is unable to form a coalition: so, either 2 forms with 1 or with 3. Consider the case in which 2 forms with 3. As $\beta \to 1$, we have: $x_1^1 = a + e - x_3^2$, $x_2^1 = 0$, $x_3^1 = x_3^2$, $x_1^2 = 0$, $x_2^2 = a - x_3^3$, $x_3^2 = x_3^3$, and

$x_1^3 = x_1^1$, $x_2^3 = 0$, $x_3^3 = a + e - x_1^1$. This implies $x_1^1 + x_3^1 = x_1^3 + x_3^3 = a + e$. We must have: $x_1^1 \geq a - d - x_2^2 = a - d - a + x_3^3 \Leftrightarrow x_1^1 \geq (a + e - d)/2$, else 1 would prefer to form a coalition with 2. Moreover, we must have $x_3^1 = x_3^3 \geq a - x_2^1 = a$, else 3 would forma coalition with 2. We therefore must have: $x_1^1 + x_3^1 \geq (3a + e - d)/2$. These inequalities are feasible only if $(3a + e - d)/2 \leq a + e$, i.e. $a \leq e + d$, impossible. Assume now that 2 forms a coalition with 1. We have:

$$\begin{aligned} x_1^1 &= a + e - x_3^2,\ x_2^1 = 0,\ x_3^1 = x_3^2 \\ x_1^2 &= x_1^3,\ x_2^2 = a - d - x_1^3,\ x_3^2 = 0 \\ x_1^3 &= x_1^1,\ x_2^3 = 0,\ x_3^3 = a + e - x_1^1 \end{aligned}$$

Implying $x_1^1 = a + e$, $x_3^1 = x_3^2 = 0$. We must have $x_3^3 \geq a - x_2^1 = a$, else 3 forms a coalition with 2, so $x_1^1 \leq e < 0$, a contradiction.

Step 3.2. It must therefore be that 1 forms a coalition with 3 and 3 forms a coalition with 2. It cannot be that 2 is unable to form a coalition: 2 can forma coalition with 3 and get $x_2^3 + (1/2)(1 - \beta)a > x_2^3$, a contradiction. Assume first that 2 forms a coalition with 3. As $\beta \to 1$, we have: $x_1^1 = a + e - x_3^2$, $x_2^1 = 0$, $x_3^1 = x_3^2$, $x_1^2 = 0$, $x_2^2 = a - x_3^3$, $x_3^2 = x_3^3$, and $x_1^3 = 0$, $x_2^3 = x_2^1$, $x_3^3 = a - x_2^1$. We have: $x_1^1 = a + e - a + x_2^1 = e$ and $x_2^2 = 0$ and $x_3^3 = a$, $x_1^3 = 0$. But then 2 could offer offer 0 to 1 and forms $\{1, 2\}$, making $a - d > 0$, a contradiction. We conclude that the only possibility is that 1 forms with 3, 3 with 2 and 2 with 1. In this case we have:

$$\begin{aligned} x_1^1 &= a + e - x_3^2,\ x_2^1 = 0,\ x_3^1 = x_3^2 \\ x_1^2 &= x_1^3,\ x_2^2 = a - d - x_1^3,\ x_3^2 = 0 \\ x_1^3 &= 0,\ x_2^3 = x_2^1,\ x_3^3 = a - x_2^1 \end{aligned}$$

This system has a unique solution with value $x_1^1 = a + e$, $x_2^2 = a - d$, $x_3^3 = a$. It is easy to verify that this is an equilibrium.

Step 4. We now show that, if $a > |e| + |d|$, then there no limit equilibrium in which 1 forms a coalition with 2. Assume first that 2 fails to form a coalition. By Step 1 it can not be that 3 forms with 1. Also by Step 1 it can not be that both 2 and 3 fail, since in this case 3 fails and is excluded by 3 and 1. So it must be that 3 forms with 2. But then we have $x_2^2 = \beta x_2^3 \leq \beta \left(a - x_3^3\right) < a - \beta x_3^3 = \widehat{x}_2^3$, where $\widehat{x}_2^3$ is what 2 can obtain by making an offer to 3. Assume then that 2 forms a coalition with 1. By Step 1, it can not be that 3 is unable to form a coalition: so, either 3 forms with 2 or with 1. Consider the case in which 3 forms with 2. As $\beta \to 1$, we have: $x_1^1 = a - d - x_2^2$, $x_2^1 = x_2^2$, $x_3^1 = 0$, $x_1^2 = x_1^3$, $x_2^2 = a - d - x_1^3$, $x_3^2 = 0$, and $x_1^3 = 0$, $x_2^3 = x_2^1$, $x_3^3 = a - x_2^1$. This implies: $x_1^1 + x_2^2 = a - d$. Moreover, we must have $x_2^2 \geq a$, else 2 would deviate and form a coalition with 3; and $x_1^1 \geq a + e - x_3^2 = a + e$, else 1 would prefer to form a coalition with 3. We therefore must have: $a-d=x_1^1+x_2^2\geq 2a+e$. These inequalities are feasible only if $a\leq |e|+|d|$, a contradiction. Assume now that 3 forms a coalition with 1. As $\beta\to 1$, we have:

$$\begin{aligned}x_1^1 &= a-d-x_2^2,\ x_2^1=x_2^2,\ x_3^1=0\\ x_1^2 &= x_1^3,\ x_2^2=a-d-x_1^3,\ x_3^2=0\\ x_1^3 &= x_1^1,\ x_2^3=0,\ x_3^3=a+e-x_1^1\end{aligned}$$

Implying $x_1^1+x_2^2=a-d$. We must have $x_1^1\geq a+e$, else 1 deviates and forms a coalition with 3. We must have: $x_2^2\geq a-x_3^3=-e+x_1^1\geq a$, else 2 would deviate and forma coalition with 3. This is feasible only if $x_1^1+x_2^2=a-d\geq 2a+e$, i.e. $a\leq |e|+|d|$, a contradiction. We must therefore have that if 1 forms a coalition with 2, then 2 forms a coalition with 3. By Step 1, it cannot be that 3 is unable to form a coalition. Assume first that 3 forms a coalition with 2. As $\beta\to 1$ we have: $x_1^1=a-d-x_2^2$, $x_2^1=x_2^2$, $x_3^1=0$, $x_1^2=0$, $x_2^2=a-x_3^3$, $x_3^2=x_3^3$, and $x_1^3=0$, $x_2^3=x_2^1$, $x_3^3=a-x_2^1$. We have: $x_2^2=a-x_3^3\leq a$. But then 2 could form a coalition with 1 offering $x_1^3=0$ and obtaining $a-d>a$.

Step 5. The only remaining possibility is that the limit equilibrium is a counterclockwise equilibrium. Consider a sequence $\beta_n>0$ with $\beta_n\to 1$ and the associated sequence of equilibria. Without loss of generality, we can assume that along the sequence 1 selects 2, 2 selects 3 and 3 selects 1 (if this is not the case, we can select a subsequence with this property). The equilibrium conditions on this sequence are given by (3) from the proof of Lemma A.6.2, Step 5. Consider now a deviation for party 3. First note that:

$$x_2^2(n)=a-\beta_n x_3^3(n)=\frac{(1-\beta_n+\beta_n^2)a-\beta_n^2 d-\beta_n e}{1+\beta_n^3},$$

$$x_2^1(n)=\frac{(\beta_n-\beta_n^2+\beta_n^3)a-\beta_n^3 d-\beta_n^2 e}{1+\beta_n^3}$$

If 3 deviates and forms a coalition with 2, s/he obtains:

$$\widehat{x}_3^3(n)=a-\beta_n x_2^1=\frac{(1-\beta_n^2+2\beta_n^3-\beta_n^4)a+\beta_n^4 d+\beta_n^3 e}{1+\beta_n^3}$$

We have a profitable deviation if:

$$\widehat{x}_3^3(n)=\frac{(1-\beta_n^2+2\beta_n^3-\beta_n^4)a+\beta_n^4 d+\beta_n^3 e}{1+\beta_n^3}>x_3^3$$

Note that

$$\widehat{x}_3^3(n)=x_3^3(n)+\frac{1-\beta_n^3}{1+\beta_n^3}\left[\beta_n\frac{1-2\beta_n+2\beta_n^2-\beta_n^3}{1-\beta_n^3}a-\beta_n d-e\right]$$

Note that: $\beta_n a\left[1-2\beta_n+2\beta_n^2-\beta_n^3\right]/(1-\beta_n^3)-\beta_n d-e\to a/3-d-e$. We have $a/3-d-e\leq 0$ only if $a\leq 3(d+e)$. If $d,e\leq 0$ this is impossible, so $a/3-d-e>0$. We conclude that with $d,e\leq 0$, 3 has a strictly positive deviation along any sequence of $\beta_n\to 1$. ■

5 The case with random proposers in C

We prove here that the choice of coalition and the formateur's payoff is the same as in (3) and (6) in the paper if we assume that a proposer in C at stage $\tau > 1$ in the intracoalitional bargaining is randomly selected among the $n(C) - (\tau - 1)$ parties in C who have not yet served as proposers with uniform probability $1/(n(C) - (\tau - 1))$. (Recall that the formateur is the first proposer at $\tau = 1$). Moreover, as $\Delta \to 0$, the coalition and the payoff of all players is the same as in Proposition 2. In the following we assume $\beta = 1$ for simplicity, the argument for $\beta < 1$ proceeds similarly (as in Lemma 1).

Let $x^*_{f,f}(C, C_f)$ be the formateur's payoff when C is proposed, but the equilibrium coalition is C_f. We must have that

$$x^*_{f,f}(C, C_f) = V(C) - \sum_{i \in C \setminus f} a^{(1)}_{f,i}(C, C_f),$$

where we are using the notation $a^{(\tau)}_{j,i}(C, C_f)$ to indicate the acceptance threshold of i at the $\tau = 1, ..., n(C)$ of bargaining when the offer is made by j. We have that:

$$\begin{aligned} x^*_{f,f}(C, C_f) &= V(C) - p \sum_{j \in C \setminus \{f\}} u_j \qquad (4) \\ &- (1-p) \sum_{k \in C \setminus \{f\}} \left[\left(\frac{1}{n(C)-1} \right) \left(\begin{array}{c} \sum_{j \in C \setminus \{f,k\}} a^{(2)}_{k,j}(C, C_f) \\ +V(C) - \sum_{j \in C \setminus \{k\}} a^{(2)}_{k,j}(C, C_f) \end{array} \right) \right] \\ &= p \left[V(C) - \sum_{j \in C \setminus f} u_j \right] + (1-p) \left(\frac{1}{n(C)-1} \right) \sum_{k \in C \setminus \{f\}} a^{(2)}_{k,f}(C, C_f) \end{aligned}$$

Note now that we must have: $a^{(n(C))}_{k,f}(C, C_f) = pu_f + (1-p)x^*_{f,f}(C_f, C_f)$ and

$$a^{(n(C)-1)}_{k,f}(C, C_f) = pu_f + p(1-p)u_f + (1-p)^2 x^*_{f,f}(C_f, C_f)$$

for all $k \in C \setminus \{f\}$. Iterating $n(C) - 2$ times, we have:

$$a^{(2)}_{j,f}(C, C_f) = p \sum\nolimits_{k=0}^{n(C)-2} (1-p)^k u_f + (1-p)^{n(C)-1} x^*_{f,f}(C_f, C_f)$$

Substituting this expression in (4), we conclude that in equilibrium we must have:

$$\begin{aligned} x^*_{f,f}(C, C_f) &= p \left[V(C) - \sum_{j \in C \setminus f} u_l \right] \qquad (5) \\ &+ \left(\frac{1-p}{n(C)-1} \right) \sum_{j \in C \setminus \{f\}} \left[\begin{array}{c} p \sum_{k=0}^{n(C)-2} (1-p)^k u_f \\ + (1-p)^{n(C)-1} x^*_{f,f}(C_f, C_f) \end{array} \right]. \qquad (6) \end{aligned}$$

It follows that:

$$x^*_{f,f}(C_f, C_f) = \max_{C \in \mathcal{C}_f} \left\{ \begin{array}{c} p\left[V(C) - \sum_{i \in C} u_i\right] \\ +p\left[1 + \sum_{k=1}^{n(C)-1}(1-p)^k\right] u_f + (1-p)^{n(C)} \cdot x^*_{f,f}(C_f, C_f) \end{array} \right\}. \quad (7)$$

Recalling that C_f is a coalition that solves (7), we immediately have that:

$$x^*_{f,f}(C_f, C_f) = u_f + \frac{p\left[V(C_f) - \sum_{i \in C_f} u_i\right]}{1 - (1-p)^{n(C_f)}}. \quad (8)$$

Assume now that we have an equilibrium in which a $C_f \neq C^*_f$, as defined in (4) in the paper (with $\beta = 1$). Then can write:

$$\begin{aligned} x^*_{f,f}(C^*_f, C_f) &= p\left[V(C^*_f) - \sum_{i \in C^*_f} u_i\right] + p\left[\sum_{k=0}^{n(C^*_f)-1}(1-p)^k\right] u_f \\ &\quad +(1-p)^{n(C^*_f)} \cdot x^*_{f,f}(C_f, C_f) \\ &= x^*_{f,f}(C_f, C_f) + \left[1 - (1-p)^{n(C^*_f)}\right] \cdot \left[\begin{array}{c} \frac{p\left[V(C^*_f) - \sum_{i \in C^*_f} u_i\right]}{1-(1-p)^{n(C^*_f)}} \\ -\frac{p\left[V(C_f) - \sum_{i \in C_f} u_i\right]}{1-(1-p)^{n(C_f)}} \end{array} \right] > x^*_{f,f}(C_f, C_f). \end{aligned}$$

Implying that indeed C_f does not solve (7) above if it does not solve (4) in the paper, a contradiction. Similarly we have that $x^*_{f,f}(C_f, C^*_f) \leq x^*_{f,f}(C^*_f, C^*_f)$ for any $C_f \in \mathcal{C}^f$: we conclude that the unique fixed-point of (4) when $\beta = 1$ is C^*_f. Given that C^*_f is selected, the derivation of the payoffs for the proposers at stages $\tau = 0, ..., n(C^*_f)$ follows the same steps as in Proposition 1.

Given the choice of C^*_f by the formateur, the payoffs of the other players are easily found. Note that the payoff of a party i when s/he is appointed as proposer is:

$$x^*_{i,i}(C^*_f, C^*_f) = u_i + \frac{p}{1 - (1-p)^{n(C^*_f)}} \left[V(C^*_f) - \sum_{l \in C^*_f} u_l\right] \quad (9)$$

independently of the stage. Let $a_i^{(k)}(C, C_f)$ be the acceptance threshold of i at stage k when i has not been proposer yet up to and including that stage. At stage $n(C^*_f) - 1$, i expect to be proposer at stage $n(C^*_f)$, so:

$$a_i^{\left(n(C^*_f)-1\right)}(C, C_f) = pu_i + (1-p)x^*_{i,i}(C^*_f, C^*_f)$$

Moreover:

$$\begin{aligned} a_i^{\left(n(C^*_f)-2\right)}(C, C_f) &= pu_i + (1-p)\left[\frac{1}{2}x^*_{i,i}(C^*_f, C^*_f) + \frac{1}{2}a_i^{\left(n(C^*_f)-1\right)}(C, C_f)\right] \\ &= pu_i\left[1 + \frac{(1-p)}{2}\right] + \sum_{l=1}^{2} \frac{(1-p)^l}{2} \cdot x^*_{i,i}(C^*_f, C^*_f) \end{aligned}$$

and:

$$a_i^{\left(n(C_f^*)-3\right)}(C,C_f) = pu_i + (1-p)\left[\frac{1}{3}x_{i,i}^*(C_f^*,C_f^*) + \frac{2}{3}a_i^{\left(n(C_f^*)-2\right)}(C,C_f)\right]$$
$$= pu_i\left[1+\frac{2(1-p)}{3}+\frac{(1-p)^2}{3}\right] + \textstyle\sum_{l=1}^{3}\frac{(1-p)^l}{3}x_{i,i}^*(C_f^*,C_f^*)$$

Assume by induction that we have defined the payoff up to j:

$$a_i^{\left(n(C_f^*)-j\right)}(C,C_f) = \left[\textstyle\sum_{l=0}^{j-1}\frac{(1-p)^l\cdot(j-l)}{j}\right]\cdot pu_i + \textstyle\sum_{l=1}^{j}\frac{(1-p)^l}{j}\cdot x_{i,i}^*(C_f^*,C_f^*)$$

Then we have:

$$a_i^{\left(n(C_f^*)-(j+1)\right)}(C,C_f) = pu_i + (1-p)\left[\frac{1}{j+1}x_{i,i}^*(C_f^*,C_f^*) + \frac{j}{j+1}a_i^{\left(n(C_f^*)-j\right)}(C,C_f)\right]$$
$$= \left[\textstyle\sum_{l=0}^{j}\frac{(1-p)^l\cdot(j+1-l)}{j+1}\right]\cdot pu_i + \textstyle\sum_{l=1}^{j+1}\frac{(1-p)^l}{j+1}\cdot x_{i,i}^*(C_f^*,C_f^*)$$

We conclude that for $\tau = 1$ we have:

$$a_i^{(1)}(C,C_f) = \left[\textstyle\sum_{l=0}^{n(C_f^*)-2}\frac{(1-p)^l\cdot\left(n(C_f^*)-1-l\right)}{n(C_f^*)-1}\right]\cdot pu_i + \textstyle\sum_{l=1}^{n(C_f^*)-1}\frac{(1-p)^l}{n(C_f^*)-1}\cdot x_{i,i}^*(C_f^*,C_f^*) \tag{10}$$

Inserting (9) in (10), we obtain:

$$a_i^{(1)}(C,C_f) = \textstyle\sum_{l=1}^{n(C_f^*)-1}\frac{(1-p)^l}{n(C_f^*)-1}\cdot\left[u_i + \frac{p}{1-(1-p)^{n(C_f^*)}}\left(V(C_f^*) - \sum_{l\in C_f^*}u_l\right)\right]$$
$$+\left[1-(1-p)^{n(C_f^*)-1} - \frac{p}{n(C_f^*)-1}\textstyle\sum_{l=0}^{n(C_f^*)-2}(1-p)^l\cdot l\right]\cdot u_i$$

Clearly we must have $x_{f,i}^*(C_f^*,C_f^*) = a_i^{(1)}(C,C_f)$. Note that As $\Delta\to 0$, we have $x_{f,j}^*(C_f^*,C_f^*)\to u_j + \frac{1}{n(C_f^*)}\left[V(C_f^*) - \sum_{k\in C_f^*}u_k\right]$ for all $j\in C_f^*$. ■

6 Externalities without transferable utility

In Section 6.4 in the paper we have argued that the baseline model can be applied to study environments with externalities on parties outside the governing coalition when we assume transferable utilities and we allow the government to tax the constituencies of the parties out of the government. When utilities are imperfectly transferable, however, it may be impossible for the coalition to reduce the utilities of the parties outside the coalition to their reservation values. In this case an agent i outside a coalition C receive a utility $v_i(C) = u_i(\eta(C))$ if no transfer is possible, or $v_i(C) = u_i(\eta(C)) + z_i(C)$, where $z_i(C)$ is the minimal transfer that can be made to i given C consistent with i being above his reservation utility u_i. Assuming here for simplicity that $z_j(C) = 0$ for $j \notin C$, a coalition C can generate a payoff vector $\mathbf{x} = (x_1, ..., x_n)$, where $x_i = u_i(\eta) + z_i$ for $i \in C$ and $x_j = v_j(C)$ for $j \notin C$, with $\eta \in P_C$, $z_i \geq -u_i(x)$ for $i \in C$ and $\sum_{l \in C} z_l \leq V(C)$.

We first note that the characterization of Proposition 1 remains unchanged by these modifications. Assume the formateur deviates to a coalition C, when the equilibrium coalition C^*. How will a player $i \in C$ evaluate an offer from the formateur? If $i \in C^*$, it knows that as soon as proposal power returns to the formateur, the formateur selects C^*, so what happens in coalitions in which i is excluded is irrelevant. If the other player i is not in C^*, it knows that as soon as proposal power returns to the formateur, C^* forms and it receives $u_i(C^*)$ (instead of zero in the case without externalities). Player i will use this value to compute its reservation utility. This however does not matter for the formateur because, as it can be seen from (5) in Section 3 of the paper, the acceptance threshold of i simplifies away from the formula characterizing the formateur's utility of selecting C, which indeed depends only on the formateur's expected utility.

What is now affected by the externalities is the analysis of Section 4, where reservation utilities are endogenous. The reason is that the reservation utility of an agent i when the formateur is f^t depends on the utility received if the formateur becomes f^{t+1}, who may select a coalition to which i does not belong. With externalities, i receives $u_i(C^{t+1})$ even if $i \notin C^{t+1}$ (without externalities, i receives zero in this case). Given this, the analysis remains qualitatively unchanged: the externalities only add additional variables in the system of equations characterizing the equilibrium).

As an example, consider a simple version of the model of Section 4 in which $\beta = 1$ and $e = d = g = 0$, so $V(\{i, j\}) = a > 0$ for all $i, j \in N$ with $i \neq j$. Now, however, assume that if coalition i, j is formed, then the remaining agent k suffers a negative externality c. The equilibrium strategies x^i_j in a counterclockwise equilibrium is characterized by a system of nine equations that can be written as:

$$x^i_i = \frac{1}{2}\left(a + x^{i+1}_i + c\right), \; x^i_{i+1} = \frac{1}{2}\left(a - x^{i+1}_i - c\right), \; x^i_{i+2} = -c$$

for $i = 1, 2, 3$ and $i + 1$ (mod 3). This system is the direct analog of (8)-(9) in Section 4 of the paper: following the same steps as in Section 4, we can show that this equilibrium always exist and yields payoffs equal to $x^i_i = \frac{2}{3}a + \frac{1}{3}c$, $x^i_{i+1} = -c$, $x^i_{i+2} = \frac{1}{3}(a - c)$, thus functions of c.

The model described in Section 2in the paper, therefore, presents a convenient framework to study legislative bargaining with externalities. In the model described above, allowing for externalities makes the analysis more complex because it adds a set of extra parameters and, likely, more cases to consider: but the problem can still be studied with the same techniques developed in Section 4. The analysis is surprisingly manageable because, in a legislature, there is one key coalition that can generate policy externalities: the government coalition. Previous work focused on more complex environments in which multiple coalitions can form at the same time (see, among others, Ray and Vohra [2001]): in such a context, the payoffs are not well defined until all coalitions are formed, thus making the analysis more difficult to analyze and often intractable.