Dynamic Collective Action and the Power of Large Numbers*

Marco Battaglini† Thomas R. Palfrey‡

May 10, 2024

Abstract

Collective action is a dynamic process where individuals in a group assess over time the benefits and costs of participating toward the success of a collective goal. Early participation improves the expectation of success and thus stimulates the subsequent participation of other individuals who might otherwise be unwilling to engage. On the other hand, a slow start can depress expectations and lead to failure for the group. Individuals have an incentive to procrastinate, not only in the hope of free riding, but also in order to observe the flow of participation by others, which allows them to better gauge whether their own participation will be useful or simply wasted. How do these phenomena affect the probability of success for a group? As the size of the group increases, will a "power of large numbers" prevail producing successful outcomes, or will a "curse of large numbers" lead to failure? In this paper, we address these questions by studying a dynamic collective action problem in which n individuals can achieve a collective goal if a share α_n of them takes a costly action (e.g., participate in a protest, join a picket line, or sign an environmental agreement). Individuals have privately known participation costs and decide over time if and when to participate. We characterize the equilibria of this game and show that under general conditions the eventual success of collective action is necessarily probabilistic. The process starts for sure, and hence there is always a positive probability of success; however, the process "gets stuck" with positive probability, in the sense that participation stops short of the goal. Equilibrium outcomes have a simple characterization in large populations: welfare converges to either full efficiency or zero as $n \to \infty$ depending on a precise condition on the rate at which α_n converges to zero. Whether success is achievable or not, delays are always irrelevant: in the limit, success is achieved either instantly or never.

JEL Classification: D71, D72, C78, C92, H41, H54 **Keywords:** Collective Action; Free Riding; Volunteering; Mechanism Design

---

*The paper has benefited from comments and suggestions by seminar audiences at Brown University, New York University, Northwestern University, Princeton University, Stanford University, and The University of Chicago. We also wish to thank Sandeep Baliga, Gary Cox, John Ferejohn, Emir Kamenica, Roger Myerson, and Andrea Prat for discussions and comments. We are responsible for any remaining shortcomings.

†Department of Economics, Cornell University, Ithaca, NY 14850. Email:battaglini@cornell.edu

‡Division of the Humanities and Social Sciences, California Institute of Technology, Mail Code 228-77, Pasadena, CA 91125. Email: trp@hss.caltech.edu.

1 Introduction

Collective action problems unfold dynamically, and require time to achieve success. Public protests often start small and, if they do not die out, reach a critical mass only gradually.[1] International agreements are initiated by small group of countries, but then require years to rally further support and collect ratifications from enough participants.[2] It is customary to leave public good and charitable-giving fund drives open for weeks or months. In these problems a common goal is achieved if and only if individual participation is sufficiently high: time allows participants to better coordinate, getting a better sense of whether the goal is achievable, which avoids wasting resources if a cause does not have enough support from early contributors.

In these situations, time is both a curse and a blessing. It is a curse because it creates incentives for individuals to defer their participation, waiting to see what others do; and this moral hazard problem is not just with respect to other players, but also against an individual's future selves. It is however also a blessing because it enables coordination and information transmission.

A significant literature has studied dynamic moral hazard games of this sort, studying the inefficiencies that arise in these environments, and even identifying conditions under which efficient allocations are possible (Schelling [1960], Fershtman and Nitzan [1991], Admati and Perry[1991], Marx and Matthews [2000], Gale [1995, 1991], Matthews [2013], Lockwood and Thomas (2002), Battaglini et al. [2014] among others). A robust lesson from this literature is that the cost of moral hazard is not so much that projects are not completed, but that they are completed with inefficient delays. Under special conditions, for example when the players are very patient and have long horizons, the delay inefficiencies typically disappear.[3]

There is however an additional important factor affecting the ability of social groups to achieve common goals that has not been fully studied in the literature, which paints a less optimistic picture. The existing literature on dynamic moral hazard focuses almost exclusively on environments with complete information, where preferences are common knowledge, so the uncertainty faced by individuals is entirely strategic: How many others will do their part? Will additional participants engage? But in environments with private information about preferences, players may become more (or less) hesitant to volunteer over time because of what they learn about the prospects for success. The relevant question, then becomes: Is there a sufficient number of committed citizens for whom it is worthwhile to participate in the collective action? With this additional consideration, there is an additional curse from time: as time progresses, players are uncertain whether delays are due to just procrastination (i.e., moral hazard), or because other players lack a willingness to participate due to high private costs (i.e., adverse selection). In these environments, players gradually acquire information on the others' preferences and willingness to contribute. Because delays from procrastination are unavoidable, this process generates a *systematic bias* toward failure. It is indeed possible that any further participation stops "cold turkey" after some histories because uncommitted individuals become too pessimistic and simply give up. How and to what extent can

[1] For example, a classic historical account of the dynamic evolution of the Great English Agricultural Uprising of 1839 is presented by Hobsbaum and Rude [1968]. More recently, Lohmann [1994] discusses the dynamic evolution of demonstrations leading to the fall of the Berlin wall in 1989.

[2] For example, the average time of ratification of the 1992 *United Nations Framework Convention on Climate Change* was 810 days. Indeed more than a year passed by the time a quarter of the countries had ratified it, and more than two years before half of the countries had ratified it (Fredrickson and Gaston [2000]).

[3] A more extensive discussion is presented in Section 1.1 below.

the passage of time still help solve the collective action problem? These questions have not been addressed and remain unanswered.

In this paper we study these and related questions in a simple but natural dynamic collective action model with private information about preferences. Formally, we model the collective action problem as a dynamic threshold contribution game with n individuals, or group members. The game takes place over a possibly infinite sequence of discrete periods. In period one, each of the n members independently and simultaneously decides whether or not to participate in the collective action. These participation decisions are binary and the associated sunk cost of participation for member i, c_i is private information and is borne immediately. If the threshold number of required participants for success, m_n, is met in period one, the game ends; each non-participant member receives the benefit v and each participant receives $v - c_i$. If the threshold is not met, the game continues to the second period, and all non-participants again must decide whether or not to join the action. The game continues indefinitely like this, with discounting, until the threshold is met at which point the game ends and each member receives the success payoff of v, discounted by the number of periods it took to reach the threshold; in addition each participant loses $-c_i$, discounted by the period when they participated. If the threshold is never met, then the game continues forever, with final payoffs equal to 0 for each non-participant, and $-c_i$ for each participant, discounted by the period when they participated.

When $m_n = 1$ the game can be interpreted as a classic war of attrition. The prize is achieving the public good without paying for it (i.e. v); at any time, a player can quit and secure a payoff $v - c_i$. When $m_n > 1$, however, the game is fundamentally different from a war of attrition. If a player quits, the payoff now is endogenous, because it depends on the externality generated by the other players completing the game. Differently from the classic war of attrition, moreover, here there is no exogenous flow cost to be paid for each period in which a player stays in the game, and it would not be natural to assume any: the cost of procrastination is already captured by the delay in receiving the public good.[4] We characterize the set of all symmetric Perfect Bayesian Equilibria (henceforth *equilibria*) of this game. We then explore the efficiency properties of the equilibria. Our analysis will focus in particular on how the inefficiencies due to delay and failure depend on the fundamentals of the model: group size; threshold level; value of the public good; the discount factor; and the distribution of costs, F.

Two basic lessons emerge from the analysis. The first is that when $m_n > 1$ collective decisions are probabilistic in all equilibria: the process starts for sure in the sense that some group members volunteer with positive probability in early periods; the final outcome, however, depends on the trajectory of participation decisions, which in turn depends on the exact realization of types. With positive probability the required threshold m_n is reached and the public good is obtained. On the other hand, the process will also get "stuck" with strictly positive probability, where additional participation ceases forever, resulting in failure.

The second lesson is that outcomes become essentially deterministic as $n \to \infty$. This however does not imply that success is guaranteed (or even possible in some equilibrium) nor that other sources of inefficiencies disappear. We indeed show that the group's success depends on the speed with which the threshold fraction of players required for success $\alpha_n = m_n/n$ converges to zero as

[4] While this is true when $m_n = 1$ as well, when $m_n = 1$ it does not have qualitative strategic implications since the game ends as soon as one player contributes. As we will explain, however, it matters a lot when $m_n > 1$.

$n \to \infty$: if α_n converges faster than the *cube root of* $1/n$, then there is a sequence of equilibria converging to the efficient collective decision with no delay; if instead it converges slower than the cube root of $1/n$, then the expected utility in the game converges to zero for all types of all players in all equilibria. We call this phenomenon the *Curse of Large Numbers* for collective action because the requirement for efficiency is very strong: even if an arbitrarily small share ε of population is required to contribute, the project is doomed for failure for large n. These lessons are in contrast with the findings of previous models, both with and without private information according to which the outcome is deterministic, and typically the efficient outcome is achieved instantly if the project is ex ante optimal and the population is large.

We start our analysis with the case of $m_n = 1$, which we call the *dynamic volunteer's dilemma.*[5] There is a unique equilibrium in which participants volunteer according to a threshold rule: at any period, t, group members with a cost below a threshold c^t volunteer and all others wait; the threshold increases over time, gradually converging to v. Similar to the analysis with complete information, a key feature of this equilibrium is that, if there is at least one player with type $c_i < v$, the collective goal is achieved for sure in finite time; all inefficiency is due to delay. The larger the population, however, the lower is the inefficiency, and in the limit the goal is achieved instantly.

We then turn to the more interesting case where more than one volunteer is needed (i.e., $m_n > 1$), which we call the *dynamic collective action problem.* Equilibria are again characterized as monotonically non-decreasing thresholds $c(h_t)$ that now may depend on the entire history h_t of contributions in the game. After each period, the players learn that no remaining type is below the threshold, so they necessarily become more pessimistic about the distribution of costs among remaining players. Now, however they may become more or less pessimistic regarding the likelihood of success over time: even though the distribution of costs types shifts up, the more participants have joined, the fewer additional ones are needed and the closer the group is to success. In this game, there can be multiple equilibria and perhaps more importantly, there is no guarantee that the project is completed, even if there are more than m_n group members with cost $c_i < v$. Participation can stop abruptly once the players become too pessimistic about the prospect of eventual success. Indeed, we show that if (as we assume in the baseline model) it is not common knowledge that all types are potentially willing to participate (i.e. $v < \bar{c}$, where $\bar{c}$ is the highest possible type), then the probability of a stoppage is positive in all equilibria. This phenomenon cannot occur in the standard war of attrition with an exogenous outside option and/or a flow cost of participation since the exogenous exit payoff would always dominate remaining in the game forever.

Even for moderate group sizes, the equilibrium characterization becomes analytically intractable: beliefs and behavior are history dependent, and there are multiple equilibria, so even numerically solving for equilibrium value functions is a daunting task. Given this intractability, it is perhaps surprising that equilibrium characterization becomes relatively simpler with large groups, i.e. cases in which n and possibly the threshold m_n grow without bound. As mentioned above, in this case there are only two possibilities: if α_n converges to zero slower than the cube root of $1/n$, then

[5] The classic volunteer's dilemma was introduced by Diekmann [1985] and refers to a static situation with complete information about preferences in which a group can achieve a collective goal if at least one member volunteers to pay a fixed cost c. It has been widely adopted as a paradigm of cooperation in economics (Bergstrom [2017]), biology (Archetti (2009), Archetti et al. [2012], Patel et al. [2018], Schneider et al. 2012), political sceince science (Goeree and Holt [2015]), neuroscience (Park et al. 2019), engineering (Roberto et al. 2011), computer science (Gasparini et al. 2020), social psychology (Darley and Latane 1968), and other fields.

in all equilibria the expected utility of every type of every player converges to zero; if instead α_n converges faster, then there is at least one equilibrium limit point in which the project is completed with probability one and the delay converges to zero, thus achieving efficiency in the limit as $n \to \infty$.

The intuition for why the cube root of $1/n$ plays a key role in the characterization can be explained intuitively. Suppose for simplicity that types are uniformly distributed in $[0, 1]$, and consider a sequence of equilibria converging to a limit in which success is achieved with no delay. A necessary condition for this to happen is that for large n the first period cutoff for participation, $c^*_{1,n}$, is sufficiently high that the at least $\alpha_n n$ members have a cost less than or equal to $c^*_{1,n}$ with probability close to 1. This, in turn, requires that $c^*_{1,n}$ converges to zero at the same rate as α_n as $n \to \infty$ (hence, $c^*_{1,n} \simeq \alpha_n$). If $c^*_{1,n}$ converged to zero slower (respectively, faster) than α_n, then the threshold for success would be passed (resp., not passed) almost surely, independently of the behavior of any individual player: thus making participation suboptimal for the indifferent type $c^*_{1,n}$, a contradiction. The cutoff type $c^*_{1,n}$, moreover, must be on the order of the expected benefit of contributing, i.e. the probability for an individual player to affect the decision in the first period, which can be shown to be proportional to $B(\alpha_n n, n, c^*_{1,n})$, the binomial probability of $\alpha_n n$ contribution out of n trials when the probability of an individual contribution is $c^*_{1,n}$.[6] Putting this all together gives:

$$\alpha_n \simeq c^*_{1,n} \simeq B(\alpha_n n, n, c^*_{1,n}) \simeq B(\alpha_n n, n, \alpha_n) \simeq \frac{1}{\sqrt{\alpha_n n}}, \tag{1}$$

where in the last step we use the well known fact that $B(\alpha_n n, n, \alpha_n)$ is on the order of $1/\sqrt{\alpha_n n}$ for large n. But condition (1) cannot hold if $\alpha_n / (1/n)^{1/3} \to \infty$, since in this case α_n converges to zero slower than $1/\sqrt{\alpha_n n}$ as $n \to \infty$. While this only establishes this rate of convergence as a necessary condition for instant success, we are able to show in Theorem 5 that there is guaranteed to exist a sequence of PBEs for which the probability of being pivotal converges to zero at the rate of $B(\alpha_n n, n, \alpha_n)$ whenever $\frac{\alpha_n}{(1/n)^{1/3}} \to 0$, which in turn implies instant success with probability 1 in the limit.

The intuition for why we get instant *failure* in all PBE when $\frac{\alpha_n}{(1/n)^{1/3}} \to \infty$ is more subtle and relies on some results in mechanism design theory. In particular, in Theorem 6 we show that for any PBE of the dynamic contribution game, we can construct a corresponding static honest and obedient (HO) mechanism (Myerson, 1982) that achieves exactly the same expected payoffs for each player type. The payoffs in the best HO mechanism therefore bounds the payoff of any PBE from above. We then use the fact that when $\frac{\alpha_n}{(1/n)^{1/3}} \to \infty$ the payoff in the best HO mechanism converges to zero (Battaglini and Palfrey [2024]) to prove the result.

The reminder of the paper is organized as follows. In the next subsection we discuss the related literature. We presents the model in Section 2. In Section 3 we study the dynamic volunteer's dilemma in which $m_m = 1$. In Section 4, we study the dynamic collective action problem in which $m_n > 1$. Section 5 is dedicated to the study of the properties of equilibria in large economies as $n \to \infty$. In Section 6 we present extensions and variations of the basic model.

[6] Indeed, a contribution may be beneficial even if the agent is not pivotal at $t = 1$, since it changes the state at which the game is played in the following periods. As we will show, in any sequence of equilibria converging to a limit efficient equilibrium, these expected benefits have only a second order effect, and can thus be ignored in this discussion.

1.1 Related literature

Our paper is most closely related to three lines of research. The first line, briefly mentioned above, studies dynamic contributions to public goods. This research has focused on settings with perfect information in which there is no uncertainty regarding the environment (say, for example, regarding the players' evaluations of the public good or their cost of contributing). The key issue in these works is the moral hazard problem faced by participants who would like the public good, but prefer for others to contribute and thus may postpone their contributions.[7] Our paper extends this analysis by considering environments where players' costs of contribution are private and heterogeneous. The key new feature of this environment is that, as time unfolds, players learn about the distribution of types and re-evaluate whether it is optimal to contribute.

The second line of research to which our work is connected is the war of attrition. Bliss and Nalebuff [1984] present a continuous time model in which a public good is achieved if at least one player volunteers for it. Players have private and heterogeneous costs of volunteering and may choose to wait hoping that other players will do it for them. In this problem, there is the usual moral hazard problem with public goods, but in addition there is uncertainty regarding the conditions under which other players will be willing to contribute. As time progresses, players update their beliefs about the cost of contributing of the remaining players. Our work extends this volunteer's dilemma framework by addressing the collective action problem with *multiple volunteers* ($m_n > 1$), and even allowing m_n to grow with n without bounds. These differences are essential to model collective action in realistic environments, since it seems natural to require multiple contributors for success in common projects with even moderately sized groups. As highlighted above, the analysis of the general collective action problem with $m_n > 1$ is qualitatively different from the volunteers dilemma. While the war of attrition has been used as leading framework in numerous important economic problems (see Alesina and Drazen [1991] for a prominent example), applications restrict the analysis to setting in which only one player needs to concede to terminate the game, as in the volunteers dilemma.[8]

The third related line of research is the work that studies public good provision when society can design optimal mechanisms (d'Aspermont and Gerard-Varet [1979], Cremer and McLean [1985], d'Aspremont, Cremer and Gerard-Varet [1990], Mailath and Postlewaite [1990], Ledyard and Palfrey [1994], Hellwig [2003], Battaglini and Palfrey [2024], among others). As in our work, this

[7]Seminal contributions are Admati and Perry [1991] and Marx and Matthews [2000]. The first paper characterizes the unique equilibrium in a game in which two players alternate contributing until the sum of contributions pass a threshold. They show that equilibrium generally implies delay and characterized conditions for efficiency, showing that they are demanding. Marx and Matthews [2000] extend the analysis to environments in which players can contribute simultaneously in each period, showing that although equilibria involve delay, the conditions for the existence of equilibria that eventually reach efficient outcomes are generally satisfied if there is a positive jump when a threshold is reached as in our environment (or if the utility for contributions is linear). Battaglini et al. [2014, 2016] show that efficient outcomes are attainable even in environments with continuous, non-linear utilities and no threshold.

[8]The only other papers we are aware of that consider extensions of the war of attrition with m multiple volunteers are Haigh and Connings [1989] and Bulow and Klemperer [1999], but these works study different economic environments, restrict attention to fixed m, and lead to very different results. The first restricts the analysis to environments with complete information. Bulow and Klemperer [1999] study an all pay auction with $m + n$ players and m prizes in which players pay a strictly positive exogenous cost κ per period to stay in the game. In the all pay auction the payoff of quitting is exogenous even with $m > 1$ since the auction allocates private goods among the other players. In this setting, they show there is a unique equilibrium in which the allocation is ex post efficient and the ex ante inefficiency converges to zero as $n \to \infty$.

literature focuses on environments with private information; but, besides restricting attention to static environments, it has a normative flavor, allowing for possibly complex communication mechanisms that require commitment power. Our analysis is positive and explicitly dynamic: we are not interested in characterizing the optimal mechanism, but in studying collective action in a natural dynamic environment in which society cannot commit to complex mechanisms for cooperation.

2 The Dynamic Collective Action Model

2.1 Model Setup

A *dynamic collective action problem* is a public good game played over a possibly infinite sequence of periods, $t = 1, 2, ..., \infty$. There is a group with n members, and each member i has a privately known participation cost c_i that is an independent draw from a commonly known cost distribution $F(c)$ with a continuous density function, $f(c)$ that is strictly positive on the interval $[0, 1]$. In each period before the game ends, each member simultaneously and independently decides whether to participate or not, a binary choice. A decision by any member i to contribute in period t is irreversible and incurs the cost c_i in the period at which i contributes. If at least m members of the group have chosen to participate up to and including in period t the game ends, and we say the group *succeeds in period* t. If fewer than m members have contributed by period t, the game continues to period $t+1$. Participation decisions are publicly observed.

Group success yields a common benefit of $v \in (0, 1)$ to all group members. Payoffs are discounted, and the discount factor is $e^{-\gamma\Delta}$, where Δ denotes the time delay between periods and $\gamma > 0$ denotes the discount rate. Hence, if the group succeeds in period t the payoff to inactive members is $ve^{-\gamma\Delta(t-1)}$ and the payoff to each member who contributed in period $\tau \leq t$ is $ve^{-\gamma\Delta(t-1)} - c_i e^{-\gamma\Delta(\tau-1)}$. If the game continues indefinitely and success is never achieved, then all members who never chose to participate receive a payoff of 0 and each member who participated in period τ receives a payoff $-c_i e^{-\gamma\Delta(\tau-1)}$. [9]

We study the set of symmetric Perfect Bayesian equilibria of this dynamic game. A strategy is a function that assigns, for each (public) history of play at period t and for each type, c, a (possibly mixed) current action to either participate or not.[10] A history at t, h_t, has two components. The first component is the sequence of participation decisions by all members in the previous periods, $t = 1, ..., t-1$. Because we are focusing on symmetric equilibria, the only relevant information is the sequence of the *number* of members choosing to participate in each period before t, $\kappa_t = (\kappa^1, ..., \kappa^{t-1})$, not the specific identities of those members who contribute. The second component of the public history is a public signal that is observed at the beginning of each period t, θ^t, which is the outcome of a single independent draw from the uniform distribution on $[0, 1]$.[11] Thus, a history at period t is denoted by $h_t = (\kappa_t, \theta_t)$, where $\theta_t = (\theta^1, ..., \theta^{t-1})$. Given a history h_t we denote by k_t the minimum number of remaining members at period t who must contribute in order for the group to achieve success. That is, $k_t = m - \sum_{\tau=1}^{t-1} \kappa^\tau$.

[9] The standard collective action problem is a one shot game and corresponds to an extreme case in our model where $\gamma\Delta = \infty$.

[10] Because participation decisions are irreversible, any individual who chooses to participate in some period τ is inactive in all future periods $t > \tau$.

[11] The public signal does not affect the characterization of equilibrium, but simplifies the existence proof.

We will characterize symmetric Perfect Bayesian equilibrium strategies of the game as a sequence of history-dependent cutpoints, $\{c(h_t\}_{t=1}^{\infty}$, whereby, in period t, following history h_t any player with a type $c \leq c(h_t)$ who has not yet participated chooses to participate.[12] Hence, in an equilibrium, the game ends in period 1 if there are at least m members with $c \in [0, c(h_1)]$, where $h_1 = (\varnothing, \theta_1)$; the game ends in period 2 if, for some $j < m$, there are exactly j members with $c \in [0, c(h_1)]$, and at least $m - j$ members with $c \in [c(h_1), c(h_2)]$, where $h_2 = (j, \theta_2)$; and so forth.

In an equilibrium, as the game progresses each remaining member's belief about the distribution of the other remaining members' types are updated simply by increasing the lower bound of the distribution of types, which we denote by $l_{h_t} = c(h_{t-1})$, with $l_{h_1} = 0$.[13] Thus an equilibrium consists of a history-dependent *cutoff strategy,* $c(h_t)$, and conditional beliefs about the distribution of remaining members, derived by Bayes rule:

$$\widetilde{F}(c; l_{h_t}) = \max\left\{0, \frac{F(c) - F(l_{h_t})}{1 - F(l_{h_t})}\right\}$$

Associated with an equilibrium are two *value functions.* For any fixed strategy and any history, h_t, we denote by $Q(h_t)$ the continuation value for a member who has previously participated (i.e., any member with $c \leq l_{h_t}$), and we denote by $V(c|h_t)$ the continuation value for a member with cost c who has not yet contributed (i.e., any member with $c > l_{h_t}$).

2.2 Characterizing Equilibrium Strategies

To solve this class of games, a key observation is that, for any given cutoff strategy each public history of play results in a new continuation game $\Gamma(h_t)$ defined by the *lower bound on the cost distribution*, l_{h_t}, and the *minimum number of contributors that are still needed for success*, k_t. This is just another collective action problem with a new group size, $n' = n - m$, a new threshold, $m' = k_t$, and a new distribution F' which is F truncated below at l_{h_t}. Hence, a solution to the original collective action involves solving for all games in this general class of collective action problems.

To characterize the equilibria of this more general class of games, we initially begin by solving the case where the continuation game is a *dynamic volunteers dilemma*, i.e., the special case of $k = 1$, and *any lower bound* l, and proceed inductively. That is, given the solutions for $k = 1$ for all $l \in [0, 1]$, and assuming we have a characterization for each $k' = 2, ..., k - 1$ for all $l_{h_t} \in [0, 1]$ we characterize the PBE for k.

3 The Dynamic Volunteer's Dilemma

3.1 Equilibrium

In this case, we solve the continuation game for a group of n members following a history h_t, at which point exactly $m - 1$ members have already contributed, so $k_t = 1$ and there are $(n - 1) - (m - 1) = n - m$ remaining uncommitted members, and the lower bound on the distribution of types is l_{h_t}.

[12] We prove in Lemmas 1 and 5 that every symmetric PBE of the game is in history-dependent cutpoint strategies.

[13] In Section 6.1 we will generalize the analysis to environments in which the players have uncertainty on the distribution of types as well. In these cases players learn over time about both the realization of types and their initial distribution.

We call this continuation game, where the group is missing exactly one contributor to succeed, *the dynamic volunteers dilemma*. We start with a preliminary result. A PBE is in *cutoff strategies* if there is a cutoff $c(h_t)$ such that any type $c \leq c(h_t)$ find it optimal to contribute, and any type $c > c(h_t)$ find it strictly optimal to wait. We have:

Lemma 1. *All PBE of a subgame starting from an history h_t in which only one contributor is missing for success are in cutoff strategies.*

Proof: See appendix. ■

We next show that the equilibrium of the continuation game, i.e., the equilibrium cutoff strategy function, $c(\cdot)$, is *uniquely determined.* The argument is as follows.

Suppose that at some history h_t we have reached a point at which the lower bound of the support is l_{h_t} and exactly $m-1$ members have already contributed, so $k_t = 1$. Denote by $V^-(c, h_t)$ the expected value for a type c who does not contribute in the current period, and $V^+(c, h_t)$ the expected value for a type c who chooses to contribute in the current period. Since $k_t = 1$, success is automatically achieved if the member contributes, so

$$V^+(c, h_t) = v - c.$$

The expression for $V^-(c, h_t)$ is slightly more complicated and depends on the continuation value if no other member contributes, which in turn depends on the current cutpoint, $c(h_t)$ (to be solved for) and the continuation value of the game if no other member contributes in the current period, in which case the lower bound of the distribution of types will change in the next period to $l_{h_{t+1}} = c(h_t)$:

$$V^-(c, h_t) = v\left[1 - \left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l_{h_t}\right)}\right)^{n-m}\right] + e^{-\gamma\Delta}\left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l_{h_t}\right)}\right)^{n-m} V(c, h_{t+1}) \tag{2}$$

We have the following observation:

Lemma 2. $l_{h_t} < v \Rightarrow c(h_t) > l_{h_t}$. *Furthermore,* $\lim_{t\to\infty} c(h_t) = v$.

Proof: See appendix ■.

In equilibrium, it must be that $V^+(c(h_t), h_t) = V^-(c(h_t), h_t)$. Furthermore, from Lemma 2, we know that $c(h_{t+1}) > c(h_t)$, so, the $c(h_t)$ type will contribute for sure in period $t+1$ if the game continues. Hence:

$$V^-(c(h_t), h_t) = v\left[1 - \left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l_{h_t}\right)}\right)^{n-m}\right] + e^{-\gamma\Delta}\left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l_{h_t}\right)}\right)^{n-m} (v - c(h_t)).$$

We conclude that the indifference condition that characterizes the equilibrium cutpoint is:

$$v - c(h_t) = v\left[1 - \left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l_{h_t}\right)}\right)^{n-m}\right] + e^{-\gamma\Delta}\left(\frac{1 - F\left(c(h_t)\right)}{1 - F\left(l_{h_t}\right)}\right)^{n-m} (v - c(h_t)) \tag{3}$$

More generally, since the game can continue for many periods without anyone contributing, (3) can be rewritten as a difference equation for all $\tau \geq t$:

$$c(h_\tau) = \left[\frac{\left(1 - e^{-\gamma\Delta}\right)\left(\frac{1-F(c(h_\tau))}{1-F(c(h_{\tau-1}))}\right)^{n-m}}{1 - e^{-\gamma\Delta}\left(\frac{1-F(c(h_\tau))}{1-F(c(h_{\tau-1}))}\right)^{n-m}}\right] v \tag{4}$$

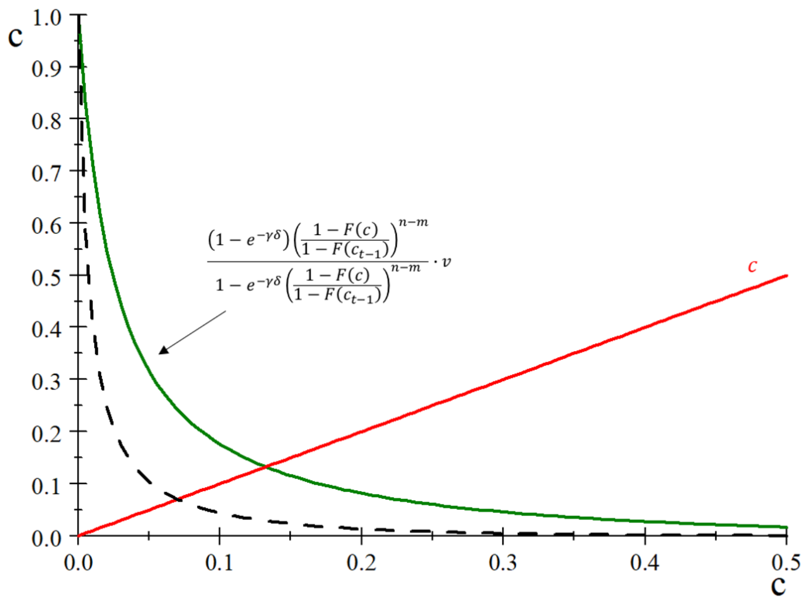

Figure 1: The equilibrium for $m = 1$. The solid and dashed curves are, respectively, the right hand side of (4) when $n = 5$ and $n = 10$, under the assumption that F is uniform, γ and Δ are such that $e^{-\gamma\Delta} = 0.95$ and $c_{t-1} = 0$.

with $c(h_{t-1}) = l_{h_t}$. Condition (4) is illustrated in Figure 1.[14] The right hand side is the opportunity cost of contributing for the cutoff type; the right hand side is the discounted net expected benefit, $c(h_\tau)$. The right hand side is a function of $c(h_\tau)$ itself, since it depends on the strategy followed by the other players, which is itself determined by the cutoff $c(h_\tau)$. The equilibrium cutoff is a fixed-point of (4). As illustrated by Figure 1, the right hand side of (4) is always decreasing, higher than c at $c = c(h_{\tau-1})$, and lower than c at $c = 1$: so there is a unique interior fixed-point $c(h_\tau) \in (c(h_{\tau-1}), 1)$. Difference equation (4) can therefore be used to mechanically construct all the equilibrium cutpoints for all h_t and the associated value functions, and thus fully characterize the unique PBE for the case of $k = 1$.

3.2 Value functions

The unique characterization of equilibrium cutpoints in the system of equations given by (4) imply two relevant value functions: (1) the equilibrium continuation value for an agent who has committed before t, given that the lower-bound of types is l_{h_t}, which we denote as $Q\left(l_{h_t}\right)$; and (2) the equilibrium continuation value of a player of type c when the lower-bound on types is l who is still uncommitted, that we denote as $V(c, l_{h_t})$. These will be useful for the full characterization when $k > 1$.

Consider first the value in period t of a member who has already committed in some previous period before t, when the lower-bound on types is $l_{h_t} = c(h_{t-1})$ and the group is still missing *exactly*

[14] In Figure 1 we assume that types are uniformly distributed, $v = 1$, $c(h_{\tau-1}) = 0$ and γ and Δ are such that $e^{-\gamma\Delta} = 0.9$.

$k = 1$ contributors for success. Note there are $n - 1 - (m - 2) = n - m + 1$ other uncommitted players. The value $Q(l_{h_t})$ for such a committed player (net of the sunk cost of contributing) when the lower-bound on the types is l_{h_t} can be written as:[15]

$$Q(l_{h_t}) = v\left(1 - B(0, n-m+1, \widetilde{F}(c(h_t)))\right) + e^{-\gamma\Delta} Q(c(h_t))\, B(0, n-m+1, \widetilde{F}(c(h_t); l_{h_t})) \quad (5)$$

where $B(0, n-m+1, x)$ is the binomial probability of 0 successes out of $n-m+1$ trials when the probability of success equals x. Using the notation $c_\tau = c(h_{t+\tau-1})$, for $\tau \geq 0$ we can use recursion to solve for $Q(l_{h_t})$ and write (5) as:

$$Q(l_{h_t}) = \sum\nolimits_{\tau=1}^{\infty} e^{-\gamma\Delta(\tau-1)} \left[\prod\nolimits_{j=1}^{\tau} \left(\frac{1-F(c_j)}{1-F(c_{j-1})}\right)^{n-m+1}\right] \left[1 - \left(\frac{1-F(c_{\tau+1})}{1-F(c_\tau)}\right)^{n-m+1}\right] v \quad (6)$$

where, by convention, $c_0 \equiv c_1 = l_{h_t}$, so $\frac{1-F(c_1)}{1-F(c_0)} = 1$. Note that (6) is defined only as a function of the primitives and the current and future cutpoints $(c_\tau)_{\tau=1}^{\infty}$ which are defined by (2).

We can also define the value of being uncommitted for a type c at $k = 1$ when the lower-bound on types is l_{h_t} and the group is missing $k = 1$ contributors for success as follows. Note that in this case there are $n - 1 - (m - 1) = n - m$ other uncommitted players. If $c \in (l_{h_t}, c(h_t)]$, we have:

$$V(c, l_{h_t}) = v - c \quad (7)$$

since the game ends when they contribute. Specifically, we have $V(c(h_t), l_{h_t}) = v - c(h_t)$.

If $c > c(h_t)$ We can define $V(c, l_{h_t})$ when $c > c(h_t)$ as follows:

$$V(c, l_{h_t}) = \left[1 - \left(\frac{1-F(c(h_t))}{1-F(l_{h_t})}\right)^{n-m}\right] v + e^{-\gamma\Delta} \left(\frac{1-F(c(h_t))}{1-F(l_{h_t})}\right)^{n-m} V(c, c(h_t)) \quad (8)$$

Using the same notation $c_\tau = c(h_{t+\tau-1})$, for $\tau \geq 0$ as in the derivation of $Q(l_{h_t})$, define $T(c)$ to as the largest τ such that $c_\tau \leq c$. Solving recursively, as with $Q(l_{h_t})$, gives:

$$\begin{aligned} V(c, l_{h_t}) \;=\; & \sum\nolimits_{\tau=1}^{T(c)-1} e^{-\gamma\Delta(\tau-1)} \cdot \left[\prod\nolimits_{j=1}^{\tau} \left(\frac{1-F(c_j)}{1-F(c_{j-1})}\right)^{n-m}\right] \left[1 - \left(\frac{1-F(c_{\tau+1})}{1-F(c_\tau)}\right)^{n-m}\right] v \\ & + e^{-\gamma\Delta\cdot(T(c)-1)} \cdot \left[\prod\nolimits_{j=1}^{T(c)} \left(\frac{1-F(c_j)}{1-F(c_{j-1})}\right)^{n-m}\right] \cdot (v - c) \end{aligned} \quad (9)$$

Note again that $V(c, l_{h_t})$ is fully determined by the cutpoints $(c_\tau)_{\tau=1}^{\infty}$.

Figure 2 qualitatively illustrates the equilibrium value function for uncommitted members, $V(c, l_{h_t})$, for $l = 0$. In the range between 0 and c_1 the value function of a player decreases with slope -1; in the range between c_1 and c_2 it decreases with slope $-(1 - \Phi_1)$; in the range between c_t and c_{t+1} it decreases with slope $-(1 - \Phi_t)$, where $\Phi_t = 1 - [1 - F(c_t)]^{n-1}$ is the *passive probability of success by period t*: this is the cumulative probability of success up to and including the current period t, for an active player who chooses not to contribute. Notice that the slopes in the figure become flatter and flatter as t increases, because Φ_t is increasing in t.[16]

[15] In writing the continuation value for the committed players as $Q(l_{h_t})$ we are slightly abusing notation, since this value is both a direct function of the lower bound of types l_{h_t}, and the history h_t, that may directly affect future cutpoints if there are multiple equilibria. We avoid writing it as $Q(l_{h_t}; h_t)$ for simplicity when it does not generate confusion. When the there is only one remaining missing volunteer, there is no loss of generality since the equilibrium is unique.

[16] The observation that φ_t is decreasing in t follows immediately from the fact that c_t is increasing in t and using equation (4).

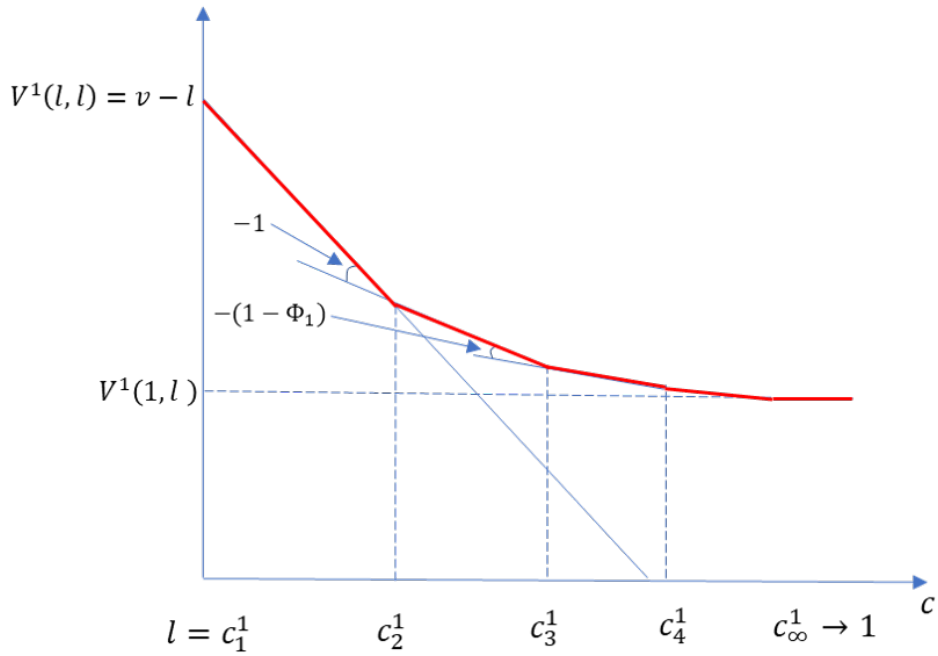

Figure 2: The equilibrium value function in the dynamic volunteer's dilemma.

The characterization of the equilibrium for the dynamic volunteers dilemma has two immediate implications. First, even when only one contributor is needed, the equilibrium is inefficient since the probability of immediate realization of the public good, $\Phi_t = 1 - [1 - F(c_1)]^{n-1}$, is strictly less than 1. Second, the distortion is not due to the fact that the project is not realized when it should be realized, but because it is realized with a delay. The project is realized by a benevolent planner if at least one player has cost that is strictly lower than v. In equilibrium the threshold for participation is always lower than v, but it gradually approaches this bound: so if there is at least one player with cost lower than v, then the project would be eventually realized.[17]

3.3 The effects of n and Δ (or γ) on the probability of success and welfare

Since the distortion depends on a delay in realization, is natural to ask whether the distortion may be mitigated by an increase in n, or a decrease in the delay costs (i.e. a reduction in either Δ or γ). For any fixed sequence of cutpoints, $\{c_t\}_{t=1}^{\infty}$, and any initial lower bound on the cost types, l, an increase in n makes it easier to achieve the target m. On the other hand, an increase in n has potentially negative equilibrium implications because the sequence of *equilibrium* cutpoints change with n. In fact, the following result shows that an increase in n leads to a uniform reduction in the equilibrium cutpoints, implying that players are individually more reluctant to contribute. Similar considerations are valid for δ: an increase in it improves welfare, since it reduces the distortions generated by delays; its increase, however, exacerbates the dynamic free rider problem, reducing

[17] These results are reminiscent of the finding in Bliss and Nalebuff [1984] who prove analogous results in a continuous time version of this model. One difference is that in Bliss and Nalebuff [1984] $v = 1$, so the project is always realized. We will discuss the connection between the apparently quite different characterizations in the two papers in Section 2.2.

the cutpoints and increasing equilibrium delays.

To study the overall effect of an increase in n, the following preliminary result will be instrumental. To make the dependence on n explicit, define $c_t(n)$ to be the cut-point with n players at period t (for some given lower bound on types, l, and discounting parameters, Δ, γ). Define as above $\Phi_t(n)$ as the cumulative probability of success up to and including the current period t, for an active player who chooses not to contribute. We say that an increase in n generates an *improvement in success probability* if $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. This implies that for a higher n the distribution is more skewed toward low values of t, which is good for the players (so an improvement). We have:

Lemma 3. *An increase in n shifts the cut-points downward so that $c_t(n) < c_t(n-1)$ for all t, but it generates an improvement in success probability.*

Proof: See appendix. ■

An increase in the size of the population induces players to be more reluctant to contribute in every period. Lemma 3 however shows that, from the point of view of any player, the increase in n more than compensates for this effect and generates an unambiguous improvement in the timing of the realization of the public good until a player decides to contribute.

The next result shows that this implies an unambiguous improvement in welfare for all players. Indeed, later where we generalize the analysis to allow for $m > 1$, we will show that the utility of all players converges to the first best v as $n \to \infty$ for any finite m (and thus for $m = 1$ as a special case as well). Let $EU_n(c)$ be the expected utility of a player of type c with n players.

Theorem 1. *An increase in n induces an increase in welfare for all types, strict for sufficiently high types: i.e., $EU_n(c) \geq EU_{n-1}(c)$ for all $c \in [0,1]$ and $EU_n(c) > EU_{n-1}(c)$ for $c_1(n)$.*

Proof. See appendix. ■

The proof of this result follows from revealed preferences and a simple inductive argument on types. The core of the argument runs as follows. If $c \leq c_1(n)$, then a type c has a payoff of $v - c$ irrespective of the total number of players. If $c \in [c_1(n), c_1(n-1)]$, then with $n-1$ players the payoff of a type c is $v - c$ and with n players the payoff of a type c is not lower than $v - c$ by revealed preferences, strictly in $(c_1(n), c_1(n-1)]$. In both cases the utility of a type $c \leq c_1(n-1)$ weakly increases with n. Assume we have proven this property for all types $c \leq c_t(n-1)$. Then this property together with Lemma 3 can be used to prove that a type c in $[c_t(n-1), c_{t+1}(n-1)]$ is strictly better off with n players than with $n-1$: even if type c does not change behavior the other players volunteer more with a higher n by Lemma 3; and if c behaves differently, then by revealed preferences this must induce an even higher utility. We can therefore extend the inductive assumption to types $c \leq c_{t+1}(n-1)$. Since $c_{t+1}(n-1) > c_t(n-1)$ and indeed we have proven above that $c_t(n-1) \to v$ as $t \to \infty$, this argument allows to show that all types $c \in [0,1]$ obtain a higher expected utility with n than with $n-1$.

Consider now the comparative statics with respect to γ and Δ. Similarly as for an increase of n, a reduction in Δ (or in γ) has an ex ante unambiguous effect on participation, leading to a downward shift in all cutpoints.[18] To make the dependence on Δ explicit, define $c_t(\Delta)$ and

[18]We only consider a change in Δ, but the results also hold for changes in γ, since the equilibrium only depends on the product of the two parameters, $\gamma\Delta$.

$\Phi_t(\Delta)$ similarly as above (for some given lower bound on types, l, and fixed values of n and γ), say that a decrease from Δ' to Δ generates an *improvement in success probability* if $\Phi_t(\Delta)$ first order stochastically dominates $\Phi_t(\Delta')$. Differently from n, now a reduction in Δ implies an deterioration in $\Phi_t(\Delta)$. This implies that although players are more patient, now success takes more time.

Lemma 4. *A decrease in Δ shifts the cut-points downward so that $c_t(\Delta) < c_t(\Delta')$ for all t and $\Delta < \Delta'$, and a downward shift in $\Phi_t(\Delta)$ in the sense first order stochastic dominance, i.e. $\Phi_t(\Delta) > \Phi_t(\Delta')$ for all t and $\Delta < \Delta'$.*

Proof: See appendix. ■

Because $\Phi_t(\Delta)$ deteriorates, the decrease in Δ has an a marginal effect on the welfare of an agent that cannot be signed: while success takes more time, the cost of delay has decreased, so these two effects go in opposite directions. The ambiguous overall effect, however, implies that contrary to what we will prove happens when $n \to \infty$, now efficiency is unattainable even in the limit as $\Delta \to 0$.[19]

Proposition 1: *For all n, v and γ, there exists $\delta > 0$ such that $\lim_{\Delta \to 0} EU_n(c) < v - \delta$ for all $c \in [0, 1]$.*

Proof. See appendix. ■

The intuitions is as follows. For any $\varepsilon > 0$, there is a strictly positive probability that all types are strictly larger than ε. As $\Delta \to 0$, the cutpoints become smaller and smaller, so no player will volunteer until it is common knowledge that all types are strictly larger than, say, $\varepsilon/2 > 0$. The equilibrium in this subgame, no player can receive a payoff larger than the payoff of the lowest type, so no player receives more than $v - \varepsilon/2$. The result follows from the fact that this subgame has a strictly positive probability.

4 The General Dynamic Collective Action Problem

If $k > 1$ contributors are required, the analysis is more complicated and we modify the notation accordingly. We denote a history by h_t^k, to indicate a period t history at which there are exactly k missing contributors.[20] Let $Q^k(l_{h_t^k})$ denote the expected utility of a committed player (net of the sunk contributing cost) at history h_t^k when the lower bound on types is $l_{h_t^k}$; and define $V^k(c, l_{h_t^k})$ to be the expected utility of an active (uncommitted) player of type c at history h_t^k.[21] As in the previous section, we say a PBE is in cutoff strategies if, for every history h_t^k, there is a critical cost $c(h_t^k)$ with the property that every type $c < c(h_t^k)$ finds it strictly optimal to contribute at h_t^k, every type $c > c(h_t^k)$ finds it strictly optimal to wait at h_t^k, and a player with type $c(h_t^k)$ is indifferent between contributing and waiting at h_t^k. We have:

[19] The fact that equilibrium converges to its efficient value as $n \to \infty$ when $m_n = 1$ will be proven as a special case of the case in which $m_n \geq 1$ and can potentially grow with n. See Theorem 7.

[20] For the case of $k = 1$, we will henceforth use the notation h_t^1.

[21] As for the case with $k = 1$, we are slightly abusing notation here to keep it simple. For a given lower bound l and number of missing volunteers k, both $Q^k(l)$ and $V^k(c, l)$ may directly depend on h_t as well in the presence of multiple equilibria since payoff irrelevant elements of the history may select the equilibrium that is played in a subgame. Omitting this information is without loss of generality when we characterize the properties of an equilibrium in a subgame for given continuation values.

Lemma 5. *All PBE of a subgame starting from an history h_t^k in which k contributors are missing for success are in cutoff strategies.*

Proof: See appendix.

In the following, we will also use the notation $c^k(l_{h_t^k})$ to denote the cutpoint $c(h_t^k)$ in order to highlight that for a given equilibrium it directly depends on the missing contributors k and the lower-bound $l_{h_t^k}$.[22] For a given history h_t^k, when it does not generate confusion, we also define as before c_t^k recursively to be $c_t^k = c^k(c_{t-1}^k)$ with an initial condition $c_0^k = l_{h_{t-j}^k}$ in case there are no contributors in the periods between from $t-j$ to $t-1$.

The previous section uniquely characterized the equilibrium for any $l \in [0,1]$ when $k = 1$, denoted here as $Q^1\left(l_{h_t^1}\right)$, $V^1\left(c, l_{h_t^1}\right)$, $c^1\left(h_t^1\right)$ for all $l_{h_t^1}$ and h_t^1, with the superscript indicating $k = 1$. We now proceed inductively on k in the following way. Assume that, for all $j = 1, ..., k-1$, the functions $Q^j\left(l_{h_t^j}\right)$, $V^j\left(c, l_{h_t^j}\right)$, $c^j\left(h_t^j\right)$ are fully defined for all $l_{h_t^j}$ and h_t^j. In the next subsection, we first show that this information enables us to characterize the cutpoints $c_t^k(l_{h_t^k})$ for all h_t^k; then, using the cutpoints $\left(c_t^j\left(l_{h_t^k}\right)\right)_{j \leq k, l_{h_t^k} \in [l_0,1]}$ the value functions $Q^k\left(l_{h_t^k}\right)$, $V^k\left(c, l_{h_t^k}\right)$ can be derived. In this way we can characterize all subgames in all histories, for all $k \leq m$ and $l \in [0,1]$. In Section 4.2 we complete the analysis of the equilibria for finite n by studying when they lead to success.

4.1 Characterization and existence

Consider first the value of an uncommitted player who contributes at h_t^k. Note that when we are missing $k > 1$ contributors, for an uncommitted player, there are $n-1-(m-k) = n-1-m+k$ other uncommitted players in the game. Hence, the value for an uncommitted player with cost c who contributes at history h_t^k is:

$$\left[V^k\right]^+(c, l_{h_t^k}) = v \sum\nolimits_{j=k-1}^{n-1-m+k} B\left(j, n-1-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right) \tag{10}$$
$$+e^{-\gamma\Delta} \sum\nolimits_{j=0}^{k-2} B\left(j, n-1-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right) Q^{k-j-1}(c^k(l_{h_t^k})) - c.$$

The first term on the right hand side of (10) is the probability the group reaches success at t times the prize v; the second term collects the probabilities that an insufficient number $j < k$ of players volunteers times the discounted expected continuation value for a contributor, $e^{-\gamma\Delta}Q^{k-j-1}(c^k(l_{h_t^k}))$; the third term is the cost of contributing.

The function $Q^{k-j-1}\left(c^k(l_{h_t^k})\right)$ used in (10) does not depend on the type of the agent, but it depends on $c^k(l_{h_t^k})$ because $c^k(l_{h_t^k})$ becomes the minimal type at $t+1$. One can simplify $\left[V^k\right]^+(c, l_{h_t^k})$ by adding and subtracting v times the probability the game does not ends at t to obtain:

$$\left[V^k\right]^+(c, l_{h_t^k}) = v - c - e^{-\gamma\Delta} \sum\nolimits_{j=0}^{k-2} \left[\left(\frac{v}{e^{-\gamma\Delta}} - Q^{k-j-1}(c^k(l_{h_t^k}))\right) B\left(j, n-1-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right)\right] \tag{11}$$

[22] In general, for a given lower bound l and number of missing volunteers k, $c^k(l_{h_t})$ also depends on h_t since h_t may determine how the others play in the presence of multiple equilibria. We omit the dependence on h_t for simplicity when it does not generate confusion.

Consider next the expected continuation payoff of an uncommitted agent of type c does not contribute at history h_t^k. Similarly as in (11), we can derive it as:

$$\left[V^k\right]^-(c, l_{h_t^k}) = v - e^{-\gamma\Delta}\sum_{j=0}^{k-1}\left[\left(\frac{v}{e^{-\gamma\Delta}} - V^{k-j}(c, c^k(l_{h_t^k}))\right)B\left(j, n-1-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right)\right] \tag{12}$$

There are now two possibilities. The first case arises if the equilibrium cutoff is a corner solution in which $c^k(l_{h_t^k}) = l_{h_t^k}$ and all types $c \geq l_{h_t^k}$ choose not to contribute. This is possible only if $\left[V^k\right]^+(l_{h_t^k}, l_{h_t^k}) \leq \left[V^k\right]^-(l_{h_t^k}, l_{h_t^k})$, which (as it can be easily verified from (11) and (12)) is equivalent to:

$$e^{-\gamma\Delta}Q^{k-1}(l_{h_t^k}) \leq l_{h_t^k} \tag{13}$$

When this condition is satisfied, even a player of type $l_{h_t^k}$ (the lowest possible at h_t^k) is unwilling to "kick the can" (i.e. contribute alone) if s/he expects no other player with type $c \geq l_{h_t^k}$ to contribute.[23] Indeed, if this player contributes alone, then the discounted benefit is $e^{-\gamma\Delta}Q^{k-1}(l_{h_t^k})$ and the cost is $l_{h_t^k}$. On the other hand, if (13) is not satisfied, then any player of type close to $l_{h_t^k}$ is willing to contribute in the hope that other players will continue contributing in history h_{t+1}^{k-1} in which the missing contributors are $k-1$. When $c^k(l_{h_t^k}) = l_{h_t^k}$, we say the equilibrium cutoff is "stuck" at history h_t^k; that is, for all practical purposes the game is over, since no uncommitted member will ever contribute in any future period, implying $V_t^k(c^k(l_{h_t^k}), l_{h_t^k}) = 0$.[24]

The other possibility is that we have an interior solution with $c^k(l_{h_t^k}) > l_{h_t^k}$. In this case the threshold $c^k(l_{h_t^k})$ is given by an indifference equation similar to the indifference condition for the volunteer's dilemma in equation (3):

$$\left[V_t^k\right]^+(c^k(l_{h_t^k}), l_{h_t^k}) = \left[V^k\right]^-(c^k(l_{h_t^k}), l_{h_t^k}) \tag{14}$$

As it can be seen from (11), all the continuation value functions used in the right hand side of this condition to define $\left[V_t^k\right]^+(c^k(l_{h_t^k}), l_{h_t^k})$ are defined by the induction step. For the left hand side, the analysis is a little more complicated. All the terms $V^{k-j}(c, c^k(l_{h_t^k}))$ are defined functions of $c^k(l_{h_t^k})$ for $j > 1$ by the induction step, but $V^k(c^k(l_{h_t^k}), c^k(l_{h_t^k}))$ is still undefined function of $c^k(l_{h_t^k})$. This is the expected utility of a type $c^k(l_{h_t^k})$ at $t+1$ in case there is no contributor at t, after the posterior is updated to the fact that there are no players with cost $c \leq c^k(l_{h_t^k})$. To define this term, note that at $t+1$, either the game stops because no type will find it optimal to contribute any longer; or the cutoff at $t+1$ is higher than the lowest type $c^k(l_{h_t^k})$, so a type $c^k(l_{h_t^k})$ finds it optimal to contribute, in which case $V^k(c^k(l_{h_t^k}), c^k(l_{h_t^k}))$ is equal to $\left[V^k\right]^+(c^k(l_{h_t^k}), c^k(l_{h_t^k}))$ as defined in (11). It follows that $V^k(c^k(l_{h_t^k}), c^k(l_{h_t^k})) = \max\left\{0, \left[V^k\right]^+(c^k(l_{h_t^k}), c^k(l_{h_t^k}))\right\}$.

For any h_t^k, the right and left hand side of (14) are defined as a functions of exclusively the cutpoints $c^k(l_{h_t^k})$ for histories in which k contributors are missing. These cutpoints can now be

[23] Strictly speaking, we have a corner solution if $\left[V^k\right]^+(l_{h_t^k}, l_{h_t^k}) < \left[V^k\right]^-(l_{h_t^k}, l_{h_t^k})$. The properties of a PBE when type $l_{h_t^k}$ is indifferent between contributing or not are the same as when the inequality is strict.

[24] From the analysis of Section 3 (or directly from (17)), we can see that an equilibrium cutoff is never stuck when $k = 1$; but when $k > 1$ it is a possibility that we will study in detail in the next subsection.

found solving (14) for any h_t^k. After some algebra, these conditions can be written as:

$$c^k(l_{h_t^k}) = e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q^{k-j-1}\left(c^k(l_{h_t^k})\right) - V^{k-j}(c^k(l_{h_t^k}), c^k(l_{h_t^k}))\right) \\ \cdot B\left(j, n-1-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right) \end{array} \right]. \tag{15}$$

where, by convention, we define $Q^0(\cdot) = v/e^{-\gamma\Delta}$. The system of equations defined in (15) characterizes $c^k(l_{h_t^k})$ in the same way as condition (4) defined the cutpoints in the case with $k=1$.[25] It also has a similar interpretation. The left hand side is the cost of contributing by the cutpoint type $c^k(l_{h_t^k})$; the right hand side is the net expected discounted utility of contributing: the difference between the utility after contributing minus the continuation in the absence of a contribution (i.e., $V^{k-j}(c^k(l_{h_t^k}), c^k(l_{h_t^k}))$). In order to prove existence of a PBE, in Theorem 3 below we prove that a fixed-point of (15) exists for any h_t^k, a step that will be discussed below.

Once we have the cutpoints when k contributors are missing, we can close the circle and define an equilibrium in all subgames up to $k = m_n$. Given the cutpoints $c^k(l_{h_t^k})$ defined above we can indeed define the expected continuation payoffs at history h_t^k to all the players as a function of $l_{h_t^k}$. The payoff to a player who has contributed in previous periods does not depend on the player's cost, c, and is given by:[26]

$$Q^k(l_{h_t^k}) = v - e^{-\gamma\Delta} \cdot \sum_{j=0}^{k-1} \left[\left(\frac{v}{e^{-\gamma\Delta}} - Q^{k-j}(c^k(l_{h_t^k}))\right) B\left(j, n-m+k, \widetilde{F}(c^k(l_{h_t^k}); l_{h_t})\right) \right]. \tag{16}$$

Using (11),(15) and (16), we can finally obtain $V^k(c, l_{h_t^k})$, which is equal to $\left[V^k\right]^+(c, l_{h_t^k})$ for $c \leq c^k(l_{h_t^k})$, and equal to $\left[V^k\right]^-(c, l_{h_t^k})$ otherwise. With this, we have all the ingredients to complete the induction argument. Using $Q^j(l_{h_t^j})$ and $V^j(c, l_{h_t^j})$ for $j \leq k$ we can now obtain $c^{k+1}(l_{h_t^{k+1}})$, and then $Q^j(l_{h_t^j})$ and $V^j(c, l_{h_t^j})$ for $j \leq k+1$. We can therefore obtain $c^j(l)$, $Q^j(l)$, and $V^j(c,l)$ for $j \leq m_n$, which fully characterizes the equilibrium. At each generic history, h_t^k, and lower bound on types $l_{h_t^k}$, the cutpoints in state k evolve according to $c_t^k = c^k(l_{h_t^k})$ where $l_{h_t^k} = c_{t-1}^k$, where $c_{t-1}^k = c^j(l_{h_{t-1}^j})$ for some $j \geq k$ and history h_{t-1}^j. The game is initialized at the history, h_1^m, where $l_{h_1^m} = 0$.

We now have a full characterization of the equilibria.

Theorem 2. *A subgame perfect equilibrium is characterized by a monotonically increasing sequence of cutpoints $c_t = c^k(l_{h_t^k})$ for $k \leq m$ and $t = 0, ..., \infty$, where $c^k(l_{h_t^k})$ is inductively defined by (11), (12), (15) and (16) as described above. For each k we have $c_{t-1} \leq c_t < v$ for each $t = 1, ..., \infty$. Furthermore if $c_t = c_{t-1}$ for any t, then $c_\tau = c_{t-1}$ for all $\tau > t$.*

Proof. Given the analysis above, we only need to prove that $c_t^k < v$. To see this, note that the right hand side of (15) is always strictly less than v since

$$e^{-\gamma\Delta}\left(Q^{k-j-1}\left(c^k(l_{h_t^k})\right) - \left[V^{k-j}\right]^+(c^k(l_{h_t^k}), c^k(l_{h_t^k}))\right) \leq e^{-\gamma\Delta}\left(Q^{k-j-1}\left(c^k(l_{h_t^k})\right)\right) < v.$$

So we have that $c^k(l_{h_t^k}) < v$ for any $l_{h_t^k} < v$. It follows that $c_t = c^k(c_{t-1}) < v$. ■

[25] One can verify that, for $k=1$ (the dynamic volunteers dilemma), equation (15) reduces to equation (4).

[26] To find (16) we first write $Q^k(l_{h_t^k})$ in terms of expected payoffs then, as we did for (11) and (12), we rewrite it by adding and subtracting v times the probability the game does not ends at t.

We complete the analysis proving the existence of a PBE.

Theorem 3. *A subgame perfect equilibrium exists.*

Proof: See appendix. ∎

To understand this result, consider (15). As discussed above, all continuation functions defining the right hand side are defined by the induction step. It is however the case that its value does not only depend on the cutpoint at t, i.e. $c^k(l_{h_t^k})$. The reason is that at $t+1$ the lower bound has moved to $c_1^k = c^k(l_{h_t^k})$, so the other players use the strategy $c_2^k = c^k(c^k(l_{h_t^k})) = c^k(c_1^k)$. If at $t+1$ we have an interior solution, $\left[V^k\right]^+(c_1^k, c_1^k)$ can be written as:

$$\left[V^k\right]^+(c_1^k, c_1^k) = v \cdot \sum_{j=k-1}^{n-1-m+k} B\left(j, n-1-m+k, \widetilde{F}(c_2^k; c_1^k\right)$$
$$+e^{-\gamma\Delta} \cdot \sum_{j=0}^{k-2} B\left(j, n-1-m+k, \widetilde{F}(c_2^k; c_1^k)\right) Q^{k-j-1}(c_2^k) - c,$$

so c_1^k depends on c_2^k. Analogously, c_2^k is itself a function of c_3^k, and so on. This implies that (15) defines c_1^k as a function of all equilibrium cutoffs that follows it along the "worst history" in which there are no additional contributors.[27] When condition (15) is required for all histories h_t^k, it defines the equilibrium cutoffs $\left(c_j^k\right)_{j=1}^{\infty}$ as a fixed-point of a correspondence that maps the sequence of cutoffs to itself. To prove existence, we proceed as follows. We first define an auxiliary truncated game in which the players give up and stop contributing if there are T attempts at which no player contributes (for some finite $T > 0$). We then show that, in this game, equilibrium cutpoints $\left(c_j^{T,k}\right)_{j=1}^{T}$ exist and are defined as fixed points of a condition similar to (15). Finally, we prove that as $T \to \infty$, these cutpoints converge to a limit $\left(c_j^k\right)_{j=1}^{\infty}$ that is an equilibrium of the original game. A key step to prove that the truncated game has a fixed-point is to show that the set of continuation values, and thus the right hand side of (15), is a non-empty, convex-, closed- valued and upper-hemicontinuous correspondence in $c^k(l_{h_t^k})$. We prove this with an inductive argument over k. To this goal, note that Section 3.1 established, by construction, the existence of a unique PBE for any lower bound on types l when only one contributor is needed. Moreover, we showed that the associated value functions $Q^1(l)$ and $V^1(c,l)$ are continuous both in l and c for any $c \geq l$. For the induction hypothesis, assume that, for all $j = 1, ..., k-1$, the set of continuation value functions $Q^j(l)$ and $V^j(c;l)$ corresponding to a PBE in the subgame is non-empty, convex-, closed- valued and upper-hemicontinuous in $l_{h_t^j}$. Using this property, we prove that the the set of continuation value functions at k, $Q^k(l)$ and $V^k(c;l)$ are non empty, convex valued and upper-hemicontinuous in l.

4.2 Participation gets stuck

In the previous section we mentioned that it is possible the equilibrium cutoff gets stuck, implying contributions stop after some history reached with positive probability in equilibrium (and this possibility has to be contemplated in the characterization). We however did not prove such an

[27] The case in which the equilibrium is not interior at $t = 1$ is simpler. In this case $\left[V^k\right]^+(c_1^k, c_1^k) = 0$, so we do not need to worry about the future cutpoints along the "worst" history with k missing volunteers.

occurrence is possible in equilibrium. Indeed, the following result shows that, while *the probability of success is always strictly positive* in every equilibrium, *the probability of getting stuck is also strictly positive* in equilibrium:

Theorem 4. *For any $n > 2$, for any $1 < m < n$, and for all $\Delta > 0$ and $\gamma \in (0, 1)$:*

1. *The probability of success is strictly positive in every equilibrium.*
2. *There are no interior equilibria. In every equilibrium there is a positive probability of reaching an effectively terminal history h_t^k with $c^k(l_{h_\tau^k}) = l_{h_t^k} < v$* for all $\tau \geq t$.

Proof: Part (1): A sufficient condition for the probability of success to be strictly positive is that $c^m(0) > 0$, since the probability of success would then be bounded below by $1-(1 - F(c^m(0)))^n > 0$. First, notice that we already proved that $c^1(0) > 0$. We now proceed by induction on the number of volunteers that are needed. Assume that for all $j = 2, ..., m - 1$, $c^j(0) > 0$ in every equilibrium and therefore $Q^j(0) > 0$ for $j = 2, ..., m - 1$. Now suppose that there is some equilibrium of the $(n, m, \gamma, \Delta, v)$ game in which the probability of success is 0. This implies that $c^m(0) = 0$ and hence $[V^m]^+ (0, 0) = 0$. But $[V^m]^+ (0, 0) \geq e^{-\gamma\Delta}Q^{m-1}(0)v > 0$, a contradiction. Hence, $c^m(0) > 0$ in every equilibrium, so the probability of success is strictly positive in all equilibria.

Part (2) Suppose by way of contradiction that there exists an equilibrium that contains a sequence $\{c_\tau^m\}_{\tau=1}^{\infty}$ with $\lim_{\tau\to\infty} c_\tau^m = c_\infty^m = v$. (If not, then the result is proved.) This is the sequence of equilibrium cutpoints along those histories where no player has contributed up to period τ. Along such a sequence, it must be that $c_\tau^m - c_{\tau-1}^m \to 0$. Hence:

$$\widetilde{F}(c_\tau^m; c_{\tau-1}^m) \to \frac{F\left(c_\tau^m\right) - F\left(c_{\tau-1}^m\right)}{1 - F\left(v\right)} \to 0$$

It follows that:

$$\begin{aligned}
\lim_{\tau\to\infty} c_\tau^m &= e^{-\gamma\Delta} \lim_{\tau\to\infty} \sum\nolimits_{j=0}^{m-1} \left[\left(Q^{h_{\tau+1}^{m-j-1}}\left(c_\tau^m\right) - V^{h_{\tau+1}^{m-j-1}}\left(c; c_\tau^m\right) \right) B\left(j, n-1, \widetilde{F}(c_\tau^m; c_{\tau-1}^m)\right) \right] \\
&\to e^{-\gamma\Delta} \left(Q^{h_{\tau+1}^{m-1}}\left(v\right) - V^{h_{\tau+1}^{m-1}}\left(c; v\right) \right) = 0 < v,
\end{aligned}$$

a contradiction. The last step follows from the fact that if the lower bound on types is v, then the expected probability that an active player contributes is zero, so $Q^{h_{\tau+1}^{m-1}}(v) = V^{h_{\tau+1}^{m-1}}(c; v) = 0$. ■

Since no player with a cost $c > v$ would ever find it optimal to contribute in any equilibrium, the highest possible probability of success achievable is $\overline{p}_n = 1 - [1 - F(v)]^n$. We use this benchmark to evaluate the performance of an equilibrium in a dynamic collective action game. We define a group to be *constrained-successful* if, in all equilibria, there will be at least m contributors by some finite date t whenever there are at least m players with cost $c < v$. That is, $\lim_{t\to\infty} c_t^k = v$, so a group is constrained-successful if the probability of success in all equilibria is $\overline{p}_n$.[28] We have:

[28] Being constrained-successful does not imply that the equilibrium is efficient. An equilibrium is efficient if the sum of the costs is lower than nv. When m is finite or when we have a threshold m_n that depends on n but such that $m_n/n \to \alpha < 1$, this condition is always satisfied for n sufficiently large. But these efficient equilibria are unachievable in a honest and obedient mechanism with no transfers.

Corollary 1. *If $m > 1$ then the group is not constrained-successful, and the probability of success is strictly less than $\overline{p}_n$ in all equilibria. In all equilibria, types with sufficiently low c contribute in early periods, but there is a positive probability of reaching an effectively terminal history h_t^k with $k < m$ and $l_{h_t^k} < v$.*

Proof: This follows immediately from Theorem 4.

The main implication of Corollary 1 is that the dynamic process of contributing is probabilistic: the process starts for sure, and players contribute with positive probability in early periods; the final outcome, however, depends on the level of participation in the periods and which depends in turn on the realization of types. With positive probability the required threshold m is reached and the public good is obtained; but with strictly positive probability the process gets "stuck". This finding shows that the general dynamic collective action problem with $m > 1$ is fundamentally different than the dynamic Volunteer's dilemma with $m = 1$ (in discrete or continuous time as in Bliss and Nalebuff [1984]). It also illustrates a sharp contrast with the multi-person war of attrition studied by Bulow and Klemperer [1999]. In all these cases, the probability the efficient allocation is reached equals 1.What the results of this section do not tell us is the size of the inefficiencies, due either to delay or failure to compete a project; and how they change as we change the environment. The probability of failure is always positive, but does it converge to zero as n increases? May this depend on the frequency of interactions Δ, or patience γ? We address these questions in the next section where we study the collective action problem in large groups.

5 Large groups

We next turn to an analysis of the properties of the PBE and welfare as $n \to \infty$. In the continuous time model of the war of attrition with $m_n = 1$ by Bliss and Nalebuff [1984], the equilibrium is asymptotically efficient as $n \to \infty$. Large numbers, therefore, eliminate the free rider problem in a collective action.[29] In this section we study the welfare properties of our more general environment in which m_n contributions are needed. In collective action problems with large groups is natural to assume that m_n is larger than one, and indeed that it grows with n. In these cases, we will show that the existence of an efficient equilibrium critically depends on the relative speed with which m_n grows with n.

As previously defined, the share of required contributors is $\alpha_n = m_n/n$. For any sequence $\{m'_n\}_{n=1}^{\infty}$, we say that m_n diverges slower than m'_n if $\lim_{n\to\infty}(m_n/m'_n) = 0$**;** m_n diverges faster than m'_n if $m_n/m'_n \to \infty$; and m_n diverges at the same rate as m'_n if $m_n/m'_n \to l$, for some finite l. Similarly, for any sequence $\{\alpha'_n\}_{n=1}^{\infty}$, we say that α_n converges to zero slower (resp., faster or at the same rate) than α'_n if $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = \infty$ (resp., $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = 0$, or $\lim_{n\to\infty}(\alpha_n/\alpha'_n) = l$ for some finite l). In the following, we maintain the assumption that $\lim_{n\to\infty} \alpha_n < v$, so achieving group success with the minimum threshold is always ex ante efficient as $n \to \infty$.

In the first best, the expected per capita utility, as $n \to \infty$, is:

$$W_n^* = v - E\left[\sum\nolimits_{j=1}^{m_n} c^{[j]}(n)/n\right] \tag{17}$$

[29] The same is true in the all pay auction model by Bulow and Klemperer [1999] in which $m > 1$, but fixed. The existence of an asymptotically efficient equilibrium is also the typical result in the literature with perfect information (see Marx and Matthews [2000] and Battaglini et al. [2014] for instance).

where $c^{[j]}(n)$ is the jth lowest cost with n samples from $F(c)$. Note that when $\alpha_n \to 0$, then $W_n^* \to v$ as $n \to \infty$; when $\alpha_n \to \alpha < v$, then we still have $W_n^* \to W^* \geq v - \alpha > 0$ as $n \to \infty$. It is also immediate to see that if $m_n > 1$, then efficiency is not guaranteed by large numbers, since equilibria with arbitrarily low probability of success are possible for any n if Δ is sufficiently large, even if m_n is constant.

The following theorem establishes an asymptotic efficiency result for large populations for the case in which α_n converges to zero sufficiently fast, showing that there is always a sequence of equilibria such that the benchmark in (17) is asymptotically achieved as $n \to \infty$. We start from a preliminary lemma that is useful to prove the result, but has independent interest. Define $c_{n,1}^m(0)$ to be the initial cutpoint at the start of the game with n members, where $t = 1$, $k = m$ and $l = 0$. The lemma shows that the share of players who volunteer in the first period, i.e. $F\left(c_{n,1}^m(0)\right)$, converges to zero at the same speed or slower than $\alpha_n = m_n/n$ if m_n diverges at infinity slower than $n^{2/3}$. For future reference, for two sequences a_n, b_n with $a_n \to 0$, $b_n \to 0$, we write $a_n \succ b_n$ if $\lim_{n\to\infty} \frac{b_n}{a_n} = 0$, and $a_n \prec b_n$ if $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$. We have:

Lemma 6. *If* $m_n \prec n^{2/3}$, *then* $\lim_{n\to\infty} F(c_{n,1}^{m_n}(0))/\left(\frac{m_n}{n}\right) > 1$.

Proof. The proof outline is as follows (see the appendix for the complete argument).

Step 1. We first find that if $m_n \prec n^{2/3}$ then $\frac{F[vB(m_n-1,n-1,\alpha_n)]}{\alpha_n} \to \infty$. To establish this property, we use Stirling's approximation of $B\left(m_n - 1, n - 1, \alpha_n\right)$ together with the fact that, in the neighborhood of 0, $F(c)$ is approximately a linear function of c with coefficient $f(0)$.

Step 2. From Theorem 4, any period 1 equilibrium cutpoint must be positive, i.e., $c_{n,1}^{m_n}(0) > 0$, and hence is given by (15) evaluated at $k = m_n$ and $l = 0$:

$$c_{1,n}^{m_n}(0) = e^{-\gamma\Delta} \sum_{j=0}^{m_n-1} \left[B\left(j, n-1, F\left(c_n^{m_n}(0)\right)\right) \left(Q^{m_n-j-1}\left(c_n^{m_n}(0)\right) - \left[V^{m_n-j}\right]^+\left(c_n^{m_n}(0), c_n^{m_n}(0)\right)\right)\right]. \tag{18}$$

The maximal fixed-point consistent with equation (18) can be bounded below by $\overline{c}_n^{m_n}(0)$ defined as follows:

$$\overline{c}_n^{m_n}(0) = \max_{c\in[0,1]} \left[c | c \leq e^{-\gamma\Delta} \sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(c\right)\right) \left[Q^{m_n-j-1}\left(c\right) - \left[V^{m_n-j}\right]^+(c, c)\right]\right]. \tag{19}$$

Let $Z_n^{m_n}(c)$ be the expression in the right hand side of the inequality in (19):

$$Z_n^{m_n}(c) = e^{-\gamma\Delta} \sum_{j=0}^{m_n-1} B\left(j, n-1, F\left(c\right)\right) \left[Q^{m_n-j-1}\left(c\right) - \left[V^{m_n-j}\right]^+(c, c)\right] \tag{20}$$

In the appendix we show that, for any period 1 cutpoint $c \geq L \cdot \alpha_n$ with $L > 1$, $Z_n^{m_n}(c)$ is bounded below by a function $z_n^{m_n}(c)$:

$$z_n^{m_n}(c) = \xi \cdot B\left(m_n - 1, n-1, F(c)\right)$$

where ξ is a positive constant.

A key step in finding the lowerbound is the observation that if $c \geq L \cdot \alpha_n$ we can ignore the payoffs obtained in all histories in which $j < m_n - 1$ players contribute: this implies that, for a lower bound, we can ignore all the terms in the summation in (20), except for the term with $j = m_n - 1$. This step would be obvious if $Q^{m_n-j-1}(c) \geq \left[V^{m_n-j}\right]^+(c, c)$ for all j, implying that all the terms are non-negative. While this property may seem intuitive, it is not automatically satisfied. The first term, $Q^{m_n-j-1}(c)$, is the expected utility of a passive player when the other players contribute knowing that the lower bound on types is c and $m_n - j - 1$ volunteers are missing. The second term is the expected utility of an active player, say player i, when the lower bound on types is c and $m_n - j$ are missing, but player i plans to contribute for sure. The second term therefore includes the cost of contribution $-c_i$, which the first term does not include. Moreover, since player i contributes for sure, in the second term, the missing volunteers for success are effectively only $m_n - j - 1$. These observations therefore would suggest that the second term should be smaller. However, in the second term, the other players volunteer as if there are $m_n - j$ missing volunteers because they do not know i's intention to contribute for sure. If the fact that they are marginally more distant from success ($m_n - j$ instead of $m_n - j - 1$ missing volunteers) induces players $-i$ to volunteer with higher probability, then the higher probability of success may compensate for the fact that player i has to contribute c_i. To bypass this complication, in the proof, we show that the expected utility of a player conditioning on fewer than $m_n - 1$ volunteers out of the remaining players is negligible relative to $B(m_n - 1, n-1, F(c))$ if $c \geq L \cdot \alpha_n$, so that (20) can be bounded below by $\xi \cdot B(m_n - 1, n-1, F(c))$ for a given positive constant $\xi \in (0,1)$.

Step 3. We conclude the proof by showing that Steps 1 and 2 imply that, for all sufficiently large n, $F(c^{m_n}_{1,n}(0)) \geq L\frac{m_n}{n}$ for some factor $L > 1$, where L may depend on $\gamma\Delta$. To this goal we use the lower bound (20) to show that (19) has a fixed point in the set $(L \cdot \alpha_n, 1)$. The logic of this step will be illustrated below using Figure 3. ■

We now use the above lemma to show that the equilibrium is successful in the first period with probability approaching 1 as $n \to \infty$ if $m_n \prec n^{2/3}$. Define P_n to be the group's ex ante probability of success, and EU_n to be the ex ante utility of a player. We have:

Theorem 5. *If α_n converges to zero faster than the cube root of $1/n$, then for all $\gamma, \Delta > 0$ there is a sequence of equilibria in which $\lim_{n\to\infty} EU_n(c) = v$ for all $c \in (0,1)$.*

The key passage in proving Theorem 5 is Step 3 of Lemma 6, where we show that:

$$\lim_{n\to\infty} F(c^{m_n}_{n,1}(0))/\left(\frac{m_n}{n}\right) \geq L > 1. \tag{21}$$

This proves that as n grows, the expected fraction of members who activate in the first period becomes strictly larger than the required threshold for success, i.e. α_n. Theorem 5 follows using this property and Chebyshev's inequality: it shows that, as $n \to \infty$, the share of players willing to contribute is larger than α_n with probability converging to 1 (and it is comprised only of types with cost arbitrarily close to zero).

The logic of Step 3 of Lemma 6 can be explained using Figure 3, where, for the purpose of this discussion we assume that the right hand side of (19) and (20) are continuous functions of c.[30] The solid curve represents $Z^{m_n}_n(c)$. Condition (19) admits a fixed point larger than $F^{-1}(L \cdot \alpha_n)$ if the solid curve evaluated at $F^{-1}(L \cdot \alpha_n)$ is above the 45^o line: in this case it admits an intersection

[30] In general, $Z^{m_n}_n(c)$ and $z^{m_n}_n(c)$ are correspondences in c, since we might have multiple equilibria, and thus multiple continuation value functions for each c. In Theorem 3 we however show that the set of continuation values defines a non empty, convex valued, upper hemicontinuous correspondence in c. These properties are sufficient for the argument outlined here to go through.

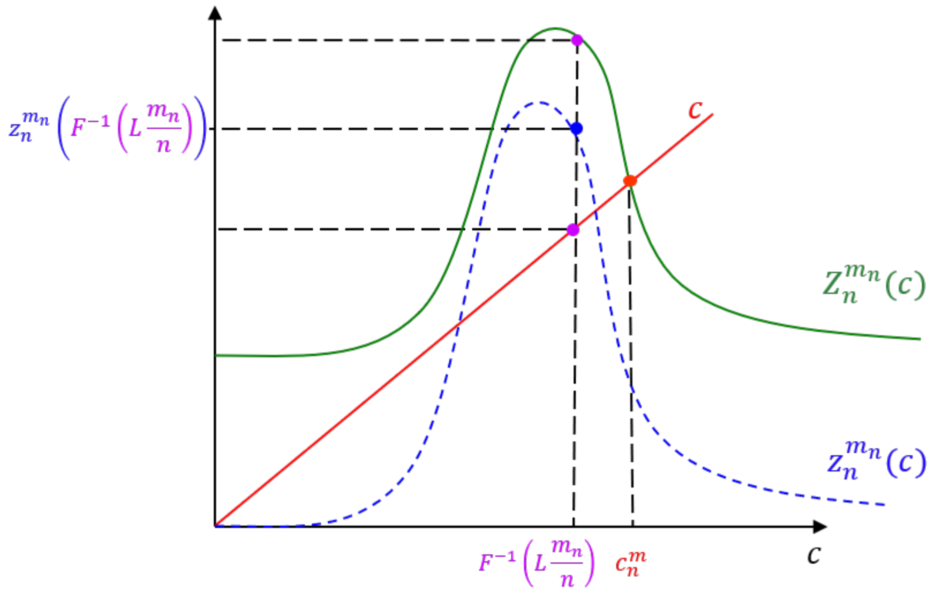

Figure 3: Condition (18) and the existence of an asymptotically efficient equilibrium.

on the right of $F^{-1}(L \cdot \alpha_n)$, since this curve converges to zero as $c \to 1$.[31] By Step 2 of Lemma 6, we can bound the solid curve below by $z_n^{m_n}(c)$, which is illustrated by the dashed curve. Hence, a sufficient condition for (21) is that $z_n^{m_n}(c)$ evaluated at $F^{-1}(L \cdot \alpha_n)$ is above the 45^o line: in this case the dashed curve intersects the 45^o line on the right of $F^{-1}(L \cdot \alpha_n)$ since this curve converges to zero as $c \to 1$, and a fortiori so does the solid line (thus implying the existence of a fixed-point $c_{n,1}^{m_n}(0) > F^{-1}(L \cdot \alpha_n)$). The key condition therefore is that $z_n^{m_n}(F^{-1}(L \cdot \alpha_n)) > F^{-1}(L \cdot \alpha_n)$, or:

$$\frac{\xi B(m_n - 1, n - 1, L \cdot \alpha_n)}{F^{-1}(L \cdot \alpha_n)} > 1 \tag{22}$$

To see that this is the case, note that using Stirling's approximation, we have:

$$\frac{\xi B(m_n - 1, n - 1, L \cdot \alpha_n)}{F^{-1}(L \cdot \alpha_n)} \simeq \frac{\xi f(0)}{L} \cdot \sqrt{\frac{1}{2\pi \cdot \alpha_n^3 (1 - \alpha_n) n}}$$

When $\alpha_n \to 0$ faster than the cube root of $\frac{1}{n}$, the right hand side of the previous expression converges to zero, and (22) is guaranteed for sufficiently large n.

We have proven above that if α_n converges to zero faster than the cube root of n, then the equilibrium payoff in the most efficient PBE converges to the efficient allocation: immediate success with probability 1. The next result shows that the cube root of n is the critical dividing threshold between complete efficiency and complete failure in large groups:

Theorem 6. *If α_n converges to zero slower than the cube root of $1/n$, then, for all $\gamma, \Delta > 0$, in every sequence PBE $\lim_{n\to\infty} EU_n(c) = 0$ for all $c \in (0, 1)$.*

[31] Note that in Figure 3, $Z_n^{m_n}(c)$ has a positive intersection at $c = 0$. This reflects the fact that the expected benefit of contributing for a $c = 0$ type is positive even if no other player contributes (recall that, in $Z_n^{m_n}(c)$, c is the cutoff adopted by the other players). In this case when $m_n > 1$, although success would be impossible in the current period, it may move the game to a state in which success will be more likely in the future.

Proof. See appendix.

To understand the strategy behind the proof of Theorem 6, we first need to recall that the equilibrium set of the dynamic collective action game characterized in Section 4 has a very complex structure, in which there is even uncertainty regarding whether the group can achieve its goal. This makes an explicit characterization difficult and indeed hopeless for large n, since the number of histories grows exponentially. The idea behind the proof of Theorem 6 is to prove that for any PBE of the *dynamic* collective action game we can define a Honest and Obedient mechanism (Myerson 1982) of a related *static* contribution game that generates the same expected payoffs.[32] This implies that the supremum of the payoffs achievable in a PBE can be bounded above by the maximal payoff achievable in the best static Honest and Obedient mechanism. As proven in Battaglini and Palfrey (2024), the per capita payoff in the best Honest and Obedient mechanism of the static game converges to zero as $n \to \infty$ if $\alpha_n / \sqrt[3]{1/n} \to \infty$. Since the payoff of a player in any PBE is nonnegative, this implies that the per capita payoff in a PBE converges to zero as well if $\alpha_n / \sqrt[3]{1/n} \to \infty$ as $n \to \infty$.

This idea can be interpreted in terms of the revelation principle. In the revelation principle it is shown that for any mechanism, there is a direct mechanism with the same payoffs for the players: the game forms associated to the mechanisms differ in terms of the players' action space; but they share the same outcome space and utility functions. Here, the dynamic game has a very different outcome space and thus different preferences over it than the corresponding static game: in the latter, the outcome is just a vector of activated players (and the associated success/failure of the common project); in the dynamic game, the outcome is a distribution over time of activated players evaluated over time.

To see why any PBE of the dynamic game defines an equivalent static HO mechanism, consider her for the sake of the argument an equilibrium in which the cutoffs $(c_\tau(\mathbf{c}))_{\tau=1}^{\infty}$ depend only on the realized types $\mathbf{c}$ (so there is no other public signal observed by the players). Even if we fix the PBE, this sequence is stochastic since it depends on the realized profile of types $\mathbf{c}$. But, for a given PBE and a given $\mathbf{c}$, it is deterministic.[33] Given a realized profile of individual costs, $\mathbf{c}$, these cutoffs define $S(\mathbf{c})$, i.e. the first period in which there are m volunteers (which may be never); $T_i(\mathbf{c})$, the first period in which player i volunteers (i.e. the first t in which $c_t(\mathbf{c}) \geq c_i$); $k_t(\mathbf{c})$ is the number of missing volunteers for success at the end of t; and $I_t(\mathbf{c})$, the set of volunteers up to and including period t.

A static mechanism can be defined as a function $\mu : [0,1]^n \to \Delta 2^I$, mapping the set of profiles of types to a distribution over the set of players who are asked to contribute. To see that a vector of cutoffs $(c_\tau(\mathbf{c}))_{\tau=1}^{\infty}$ define such a mechanism, consider the following multi-step algorithm. When profile $\mathbf{c}$ is reported, in Step 1 all individuals with a type below $c_1(\mathbf{c}) = c^m(l)$ are asked to volunteer (i.e., the set $I_1(\mathbf{c})$). If there are at least m such individuals, i.e., $k_1(\mathbf{c}) = 0$ and $S(\mathbf{c}) = 1$, then the public good is provided and the algorithm stops without proceeding to Step 2. In this case, $S(\mathbf{c}) = 1$ and $\mu^{DYN}_{I_1(\mathbf{c})}(\mathbf{c}) = 1$ (we denote by $\mu^{DYN}_g(\mathbf{c})$ the probability a group g is asked to volunteer when the profile is $\mathbf{c}$). If $k_1(\mathbf{c}) > 0$, i.e., $S(\mathbf{c}) > 1$, then with probability $1 - e^{-\gamma\Delta}$ the algorithm also

[32] The related static contribution game is the same contribution game as the game described in Section 2.1, but there is only one period, after which if success was not reached no player can no longer contribute.

[33] In the presence of public signals, the sequence may also depend on the realization of the signals, which determines the continuation equilibrium that is chosen. See the proof of Theorem 6 in the appendix for details.

stops without proceeding to Step 2 (and the public good is not provided). In this case, $S(\mathbf{c}) > 1$ and $\mu^{DYN}_{I_1(\mathbf{c})}(\mathbf{c}) = 1 - e^{-\gamma\Delta}$. With probability $e^{-\gamma\Delta}$, instead, the algorithm proceeds to Step 2. In Step 2, all individuals with a type in the interval $(c_1(\mathbf{c}), c_2(\mathbf{c})]$ where $c_2(\mathbf{c}) = c^{k_1(\mathbf{c})}(c_1(\mathbf{c}))$ are also asked to volunteer and the process continues. In general, at any Step t at which the algorithm has not yet stopped, individuals with a type in the interval $(c_{t-1}(\mathbf{c}), c^{k_{t-1}(\mathbf{c})}(c_{t-1}(\mathbf{c}))]$ are asked to volunteer. If there are at least $k_{t-1}(\mathbf{c})$ such individuals, i.e., $k_t(\mathbf{c}) = 0$ and $S(\mathbf{c}) = t$, then the public good is provided in Step t and the algorithm stops selecting $I_t(\mathbf{c})$ without proceeding to step $t+1$. If $k_t(\mathbf{c}) > 0$, i.e., $S(\mathbf{c}) > t$, then with probability $1 - e^{-\gamma\Delta}$ the algorithm also stops without proceeding to step $t+1$ (and the public good is not provided), and with probability $e^{-\gamma\Delta}$ the algorithm proceeds to step $t+1$.

The algorithm described above defines the following static mechanism:

$$\mu^{DYN}_g(\mathbf{c}) = \begin{cases} \sum_{\{\tau | I_\tau(\mathbf{c}) = g\}} \left(1 - e^{-\gamma\Delta}\right) e^{-\gamma\Delta(\tau-1)} & |g| < m \\ e^{-\gamma\Delta(S(\mathbf{c})-1)} & \text{if } |g| \geq m \text{ and } g = I_{S(\mathbf{c})}(\mathbf{c}) \\ 0 & \text{else} \end{cases} \tag{23}$$

which mimics the discounting in the dynamic game by randomly stopping the algorithm with probability $1 - e^{-\gamma\Delta}$ after any step at which the threshold m has not yet been achieved.

The proof is completed by showing that $\mu^{DYN}_g(\mathbf{c})$ is Honest and Obedient mechanism. This fact is intuitive. By construction, the static mechanism asks a player to contribute if and only if the player is in an event that mimics an history in which the player finds it optimal to contribute in the dynamic game. The event "mimics" such an history in the sense that conditioning on such an event, the player has a the same posterior on the other players types as after such an history. It follows therefore that if the static mechanism is not honest and obedient, then we would have a deviation in the PBE, a contradiction.

6 Extensions and variations

6.1 Aggregate uncertainty and learning

In the previous analysis we assumed the players' types are i.i.d. In such environments, as time progresses players update their beliefs about the types of the other players because they know that the remaining active players must have a type higher than the previous equilibrium cutoffs. Yet, they do not learn anything new about the original environment, since the distribution of the other players' types is common knowledge. It is natural to consider scenarios in which players are also ex ante uncertain about the environment, for example about the shape of the distribution of types. In these environments, as time progresses, players also learn about the shape of the distribution of types and this learning also depends on equilibrium strategies.

To illustrate how the analysis can be generalized to this more complex case, we characterize here the equilibrium in the volunteer's dilemma with $m_n = 1$ as in Section 3. The analysis can extended in similar ways to the case with $m_n > 1$. We assume for simplicity that there are two states of nature $\vartheta = H, L$; and that the distribution of types is $F_\vartheta(c)$ in state ϑ, with density $f_\vartheta(c)$ and $F_H(c)$ first order stochastically dominating $F_L(c)$. In this environment, at $t = 1$, a player's belief that the state is H at the beginning of the first period is necessarily function of their type

$\pi^1(c)$ since for any initial common prior π^0, they would update after observing their type. We assume here that $\pi^1(c)$ is increasing and continuous in c.[34]

Consider a threshold equilibrium with cutoffs $(\dot{c_t})_{t=0}^{\infty}$ and $\dot{c_t} > \dot{c}_{t-1}$ as in Section 3, such that at t all types $c \leq \dot{c_t}$ contribute, and types $c > \dot{c_t}$ wait. Given these cutoffs and belief $\pi^{t-1}(c)$ at $t-1$, the belief at the beginning of period $t > 1$ for a type c is given by:

$$\pi^t(c) = \left[1 + \frac{1-\pi^{t-1}(c)}{\pi^{t-1}(c)} \frac{\left(\frac{1-F_H\left(\dot{c}_{t-1}\right)}{1-F_H\left(\dot{c}_{t-2}\right)}\right)}{\left(\frac{1-F_L\left(\dot{c}_{t-1}\right)}{1-F_L\left(\dot{c}_{t-2}\right)}\right)}\right]^{-1} \tag{24}$$

Note that, for any t, if $\pi^{t-1}(c)$ is continuous and increasing in c, then $\pi^t(c)$ is continuous and increasing in c as well since $\frac{1-\pi^{t-1}(c)}{\pi^{t-1}(c)}$ is continuous and decreasing in c.

Given the posterior at t, $\pi^t(c)$, and the equilibrium lower bound $\dot{c}_{t-1}$, an argument analogous to the argument leading to (3) in Section 3, gives us the cutoff at t as the fixed-point of:

$$v - \dot{c_t} = \sum\nolimits_{\theta=H,L} \pi_\theta^t(\dot{c_t}) \left\{ v \left[1 - \left(\frac{1-F_\theta\left(\dot{c_t}\right)}{1-F_\theta\left(\dot{c}_{t-1}\right)}\right)^{n-1}\right] + e^{-\gamma\Delta} \left(\frac{1-F_\theta\left(\dot{c_t}\right)}{1-F_\theta\left(\dot{c}_{t-1}\right)}\right)^{n-1} (v - \dot{c_t}) \right\} \tag{25}$$

note that now $\dot{c_t}$ does not only affect $\frac{1-F_\theta(\dot{c_t})}{1-F_\theta\left(\dot{c}_{t-1}\right)}$ in the right hand side of (26); but also the prior probabilities $\pi_\theta^t(\dot{c_t})$ at t with which the shape of the distribution is evaluated: this because all players of different types have different posteriors at each history of the game, since they start from heterogeneous priors $\pi^1(c)$. After some algebra, we have:

$$\dot{c_t} = \frac{\left(1-e^{-\gamma\Delta}\right) \cdot \sum_{\theta=H,L} \pi_\theta^t(\dot{c_t}) \left(\frac{1-F_\theta(\dot{c_t})}{1-F_\theta\left(\dot{c}_{t-1}\right)}\right)^{n-1}}{1 - e^{-\gamma\Delta} \sum_{\theta=H,L} \pi_\theta^t(\dot{c_t}) \left(\frac{1-F_\theta(\dot{c_t})}{1-F_\theta\left(\dot{c}_{t-1}\right)}\right)^{n-1}} v \equiv G(\dot{c_t}) \tag{26}$$

Condition (26) alone is no longer sufficient to characterize the equilibrium. The equilibrium now is determined by the system of difference equations (24) and (26): $\pi^1(c)$ and the initial lower bound $\dot{c}_0$ determine $\dot{c}_1$; $\pi^1(c)$, $\dot{c}_0$ and $\dot{c}_1$ determine $\pi^2(c)$; $\dot{c}_1$, $\dot{c}_2$ and $\pi^2(c)$ determine $\dot{c}_3$; and so on so forth for any $t > 0$.

Condition (26) can be used to show that a cutoff equilibrium has similar properties to the equilibria as in Section 3. To see this, consider the right hand side of (26), $G(\dot{c_t})$. This function is continuous in $\dot{c_t}$ and has the properties that $G(\dot{c}_{t-1}) = v$ and $G(1) = 0$; hence it has a fixed point $\dot{c_t} > \dot{c}_{t-1}$ for any $\dot{c}_{t-1}$ and $t-1$. We can moreover verify that $\dot{c_t} > \dot{c}_{t-1}$ and $\dot{c_t} \to v$. For this, assume by way of contradiction that $\lim_{t\to\infty} \dot{c_t} = \dot{c}_\infty < v$. Then we would still have $\lim_{t\to\infty} \pi^t(\dot{c_t}) = \pi^t(\dot{c}_\infty)$ and $\sum_{\theta=H,L} \pi_\theta^t(\dot{c}_\infty) = 1$. It follows that:

$$\lim_{t\to\infty} \dot{c_t} = \frac{\left(1-e^{-\gamma\Delta}\right) \cdot \sum_{\theta=H,L} \lim_{t\to\infty} \pi^t(\dot{c_t}) \left(\frac{1-F_\theta(\dot{c_t})}{1-F_\theta\left(\dot{c}_{t-1}\right)}\right)^{n-1}}{1 - e^{-\gamma\Delta} \sum_{\theta=H,L} \lim_{t\to\infty} \pi^t(\dot{c_t}) \left(\frac{1-F_\theta(\dot{c_t})}{1-F_\theta\left(\dot{c}_{t-1}\right)}\right)^{n-1}} v = v,$$

[34] For an event $E = [c-\varepsilon, c+\varepsilon]$, we have $Pr(H;E) = \frac{\pi^0 \Delta F_H(E)}{\pi^0 \Delta F_H(E) + (1-\pi^0)\Delta F_L(E)}$, where we define $\Delta F_\vartheta(E) = [F_\vartheta(c+\varepsilon) - F_\vartheta(c-\varepsilon)]$. Taking the limit as $\varepsilon \to 0$, we have that $Pr(H;c) = \frac{\pi^0 f_H(c)}{\pi^0 f_H(c) + (1-\pi^0) f_L(c)}$ is increasing if $f_H(c)/f_L(c)$ is increasing in c.

a contradiction.

When the players learn about the distribution of types, however, an additional complication may arise regarding whether an equilibrium is necessarily in cutoff strategies. To see this point, consider (25) which characterizes the indifferent type $\dot{c}_t$. For a type $c < \dot{c}_t$, the left hand side is $v - c$, so it decreases linearly with a slope of -1. The right hand side now is:

$$\sum\nolimits_{\theta=H,L} \pi_\theta^t(c) \left\{ v \left[1 - \left(\frac{1 - F_\theta(\dot{c}_t)}{1 - F_\theta(\dot{c}_{t-1})} \right)^{n-1} \right] + e^{-\gamma\Delta} \left(\frac{1 - F_\theta(\dot{c}_t)}{1 - F_\theta(\dot{c}_{t-1})} \right)^{n-1} (v - c) \right\}$$

where we note that c enters only in the posterior $\pi_\theta^t(c)$ and in the last term $v - c$, since $\dot{c}_t$ is the strategy used by the other players. When $f_H(c)/f_L(c)$ (and therefore a fortiori $\pi^1(c)$ and $\pi^t(c)$) does not increase too sharply in c, then this term certainly declines in c at a rate slower than -1, so types $c < \dot{c}_t$ find it optimal to abstain and types $c > \dot{c}_t$ to contribute (just as in the case with no aggregate learning). But when $f_H(c)/f_L(c)$ can change sharply (as for example when the support of costs changes with the state, so that the posterior is discontinuous in c), then the equilibrium may not be in cutoff strategies. It is interesting that even in the simplest case with $m = 1$ we might have these complications.

6.2 Non-stationary environments

There are applications for collective action problems in which it seems natural to assume that the value of a group's success changes over time. In environmental problems, for example, the consequences of not solving the collective problem (i.e. failing to succeed in collective action) become more severe over time. In this section we show how non-stationarities can be easily incorporated in the analysis and lead to new insights.

Assume that if the group does not obtain the common goal at $t - 1$, then at the beginning of t each player suffers a loss of Z^t for $Z > 1$. The loss from not obtaining the common goal (say closing the ozone hole) grows exponentially over time.[35] The equilibrium condition now becomes:

$$v - \ddot{c}_t - Z^t = v \left[1 - \left(\frac{1 - F(\ddot{c}_t)}{1 - F(\ddot{c}_{t-1})} \right)^{n-1} \right] + e^{-\gamma\Delta} \left(\frac{1 - F(\ddot{c}_t)}{1 - F(\ddot{c}_{t-1})} \right)^{n-1} (v - \ddot{c}_t - Z^{t+1}) - Z^t$$

At time t, the cost Z^t is sunk, and thus irrelevant for the decision. But the cost at $t+1$ still matters, since if the group is successful at t, it can avoid the cost Z^{t+1}. As in the previous sections, the left hand side is the utility from contributing, the right hand side is the utility for not contributing. In both cases, a players suffers a cost Z^t, which then can be simplified. In the utility for not contributing we now have Z^{t+1} times the probability that none of the other players contributes. The loss Z^{t+1} does not affect the decision at $t + 1$, when it is a suck cost, but it affects the decision at t. This condition can be rewritten as:

[35] There are of course other ways to introduce non-stationary elements in the model (for example we could have assumed that the distribution of c or v changes over time). We chose to model non-stationarity as above because it seems it better captures the phenomenon described in the example of environmental protection described above.

$$c_t^{\cdot\cdot} = \left[\frac{\left(1 - e^{-\gamma\Delta}(1 - \frac{Z^{t+1}}{v})\right)\left(\frac{1-F(c_t^{\cdot\cdot})}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}}{1 - e^{-\gamma\Delta}\left(\frac{1-F(c_t^{\cdot\cdot})}{1-F\left(c_{t-1}^{\cdot\cdot}\right)}\right)^{n-1}} \right] v \tag{27}$$

For a given $c_{t-1}^{\cdot\cdot}$ and $c_t^{\cdot\cdot}$, the right hand side increases in Z and t. The monotonic worsening of the environment reduces the utility of not contributing and leads to a lower utility of not contributing. The amount of the increase, however is endogenous and depends in the equilibrium $c_t^{\cdot\cdot}$. Interestingly, even for an arbitrarily large Z^t, the group is unable to guarantee success. This can be seen from (27), since $c_t^{\cdot\cdot} > c_{t-1}^{\cdot\cdot}$ but $c_t^{\cdot\cdot} < 1$ for all Z^{t+1}.

6.3 Dynamic collective action with high value ($v \geq 1$)

Up to this point we assumed $v < 1$. If $v \geq 1$, *then it is common knowledge that all players would willingly participate if they are pivotal.* This has a number of implications, which we explain here. First, there exist asymmetric equilibria that achieve success instantly: in equilibrium, at $t = 1$ a subset of exactly m members participates, regardless of their private cost, and the remaining $n - m$ members free ride. Of course, this requires some coordination device among the players. If such coordination devices are not readily available, then we are back to characterizing the symmetric PBE of the game. In this high value case, results are not entirely negative.

For symmetric PBE, we have the following results. We extend Theorem 4 to the high-value case as follows:

Theorem 4′. *If $v \geq 1$ then for all $n > 2$, for all $1 < m < n$, and for all $\Delta > 0$ and $\gamma \in (0, 1)$:*

1. *The probability of success is strictly positive in every equilibrium.*
2. *There exists a minimum value threshold, $1 < v^*(n, m, \gamma, \Delta) < \infty$, such that all equilibria are interior if and only if $v > v^*(n, m, \gamma, \Delta)$, and success is achieved with probability* 1.
3. *For all $v \in [1, v^*(n, m, \gamma, \Delta)]$, there is at least one equilibrium in which there is a positive probability of reaching an effectively terminal history h_t^k with $c^k(l_{h_\tau^k}) = l_{h_t^k} < v$* for all $\tau \geq t$.

Proof: See appendix. ■

The proof of part (1) is the same as in Theorem 4. Part (2) is proved by induction. A detailed proof is in the appendix, which we sketch here. When $m = 1$, all equilibria are interior by Lemma 2 for all v, (2) therefore holds for $m = 1$. The properties of the interior equilibrium when $k = 1$ moreover guarantee that we have $e^{-\gamma\Delta}Q^1(l) - l > 0$ for any $l \in [0, \min\{v, 1\}]$. For the induction hypothesis, assume that for all $j = 1, ..., k - 1$ there exists a $v_{k-1}^*(n, j, \gamma, \Delta) < \infty$ such that $l \in [0, \min\{v, 1\}]$ implies that in every equilibrium, $c^j(l) > l$ and $e^{-\gamma\Delta}Q^j(l) - l > 0$, if and only if $v > v_{k-1}^*(n, j, \gamma, \Delta)$. The next step of the proof is to show that the induction hypothesis implies the existence of a $v_k^*(n, k, \gamma, \Delta) > 1$ such that $l \in [0, \min\{v, 1\}]$ implies that in every equilibrium, $c^k(l) > l$ and $e^{-\gamma\Delta}Q^k(l) - l > 0$ if and only if $v > v_k^*(n, k, \gamma, \Delta)$. This requires some care since $Q^k(l)$ is typically a complicated function of l for $k > 1$. This argument allows us to conclude that for any $k \leq m$, a player with type close to l finds it optimal to contribute, even if s/he expects all other active players not to contribute.

Part (3) of Theorem $5'$ can be seen as the residual case and follows as a corollary to Part (2). If $v \le v^*(n, m, \gamma, \Delta)$, then there is a history with positive probability at which if a player expects no other player to contribute, then s/he does not find it optimal to contribute as well: thus we have an equilibrium in which players stop contributing. This in itself, however, does not imply that there is no interior equilibrium in which success is eventually achieved since the payers' decisions to contribute may be strategic complements, implying that we could have additional equilibria in which participation is stimulated by the expectation that other players contribute with positive probability. As shown in Theorem 4, however, this is impossible if $v < 1$.

Theorems 4 and $4'$ highlight the fact that, except when v is very high, there is uncertainty regarding whether the group can get stuck at an effectively terminal history where no player is willing to contribute anymore, or, alternatively, the cutpoints continually increase in all periods. However it leaves open two issues. First, it is possible that for all $k < m$ we have $c_t^k > c_{t-1}^k$, but $c_t^k \to c_\infty^k < 1$. In this case, the equilibrium is interior but the project may still remain unrealized even if all types are below v. The second question concerns the size of $v^*(n, m, \gamma, \Delta)$. Should we expect $v^*(n, m, \gamma, \Delta)$ to be close to 1, at least when players are patient (i.e., $\gamma, \Delta \to 0$)?

The following proposition addresses the first issue.

Proposition 2. *If $v > v^*(n, m, \gamma, \Delta)$, then the group is constrained-successful, i.e., $\overline{p}_n = 1 - [1 - F(v)]^n = 1$ in all equilibria. If $v \in [1, v^*(n, m, \gamma, \Delta)]$, then the group is constrained successful in any interior equilibrium.*

Proof: The proof shows that $c_t^k \to c_\infty^k = v$ in any interior equilibrium. See appendix for details.

The next result addresses the second issue, the size of $v^*(n, m, \gamma, \Delta)$. We have:

Proposition 3. *For any $n > 2$, for any $1 < m < n$, and for all $\Delta > 0$ and $\gamma \in (0, 1)$: $v^*(n, m, \gamma, \Delta) \ge 2$ for any n, m, γ and Δ.*

Proof: See appendix for details.

This result tells us that even if $v \in [1, 2]$, where it is common knowledge that all players desire the public good even if their participation is required to get it, the dynamic process of participation is probabilistic, and a strictly positive probability of failure is unavoidable. This is true even if the group is large, the threshold is small, and even if the players are arbitrarily patient. The inability of a group to reach success does not depend on the frequencies of interaction Δ, so it remains true even in the limit as $\Delta \to 0$.

For large groups $v^*(n, m, \gamma, \Delta)$ grows without bound. Specifically, we have:

Proposition 4. *If $m_n > 1$, then $\lim_{\Delta \to \infty} v^*(n, m_n, \gamma, \Delta) = \infty$.*

Proof: See appendix for details.

It is important to note that Proposition 4 does not preclude the possibility that even if Δ is large, there can be efficient or approximately efficient equilibria in the limit. In particular, it is easy to see that Theorem 5 holds for all $v > 0$, including for the high value case.

6.4 On the time horizon and the effectiveness of imposed deadlines

Theorem 6 shows that, while the ex ante probability if success is strictly optimal for any n, the limit probability of success always converges to zero, except if $\alpha_n \to 0$ sufficiently fast. This suggests the question of whether there are simple modifications in the strategic interaction of the players that may avert this "curse of large numbers" for the collective action problem. We leave the general problem of studying the optimal dynamic mechanism for future research, but focus here on the discussion of simple rules that the group could adopt hoping to improve the performance: *deadlines*, where the group commits to terminate the volunteering game if the goal is not reached by some specified finite period T (if such a commitment power is granted to the group). While terminating the game at T may be suboptimal once period T arrives, commitment to such a rule may be beneficial if it stimulates more volunteering in periods $\tau = 1, ...T$. Intuitively, in the dynamic collective action problem players have a free rider problem not only with respect to other participants, but to the future selves of all participants (including themselves); imposing a terminal period for contributions could, in principle, limit this problem.

The following result shows that in large groups, deadlines may have an especially undesirable side effect by generating a unique equilibrium in which participation is exactly zero, so the group does not even try to achieve the goal. For a sufficiently high n, therefore, a deadline of T periods is strictly suboptimal since it generates a payoff of exactly zero, while by Theorem 4 we know that in the unbounded game the probability of success is always strictly positive, thus the expected payoff is strictly positive.

Proposition 5. *Assume $m_n = \alpha n$ for some $\alpha < 1$. For any finite deadline T, participation is exactly zero if n is sufficiently large.*

Proof. See appendix.

The threshold n^*_T on n such that participation is zero for $n > n^*_T$ may depend on the other parameters of the game. Proposition 5 says that as $n \to \infty$, the minimal deadline consistent with positive participation must also diverge at infinity; for any finite deadline T, positive participation is inconsistent with sufficiently large groups.

To see the intuition of this result, consider first the case in which $T = 1$. In this case the equilibrium cutpoint is determined by the equation:[36]

$$c_n^{1,1} = vB\left(m_n - 1, n - 1 - m_n, F\left(c_n^{1,1}\right)\right) \tag{28}$$

The left hand side is the cost of contributing for the marginal type; the right hand side is the expected benefit, that is v times the probability of being pivotal. An equilibrium cutoff is a fixed point of this equation.

Figure 4 illustrates it: the 45^o line in red line is the left hand side of (28); the black lines are the right hand side of (28), for different values of n.[37] As it can be seen, as n increases the right hand side shifts down; when n is 100 or larger, the curve is uniformly below the 45^o degree line for any $p > 0$: this implies that no type except $c = 0$ is willing to volunteer. In Proposition 5 we indeed show that when the number of missing volunteers m_n grows at the speed of n (as when $m_n = \alpha n$ for some $\alpha < 1$) and $T = 1$ (or there only one period left before termination), then there $n^{(1)}$ such that for $n > n^{(1)}$ there is no equilibrium in which players with $c > 0$ contribute.

When $T > 1$, we can show that this phenomenon generalizes with an inductive argument, but there are some complications. Consider, for simplicity, $T = 2$. From the discussion above, we

[36]We denote $c_n^{\tau,T}$ to be the cutoff at period τ in a model with a deadline with T periods.

[37]In the figure c is uniformly distributed, $v = 1$, and $n = 30$ (the solid line),50 (the intermediate dashed line) and 100 (the lower dashed line).

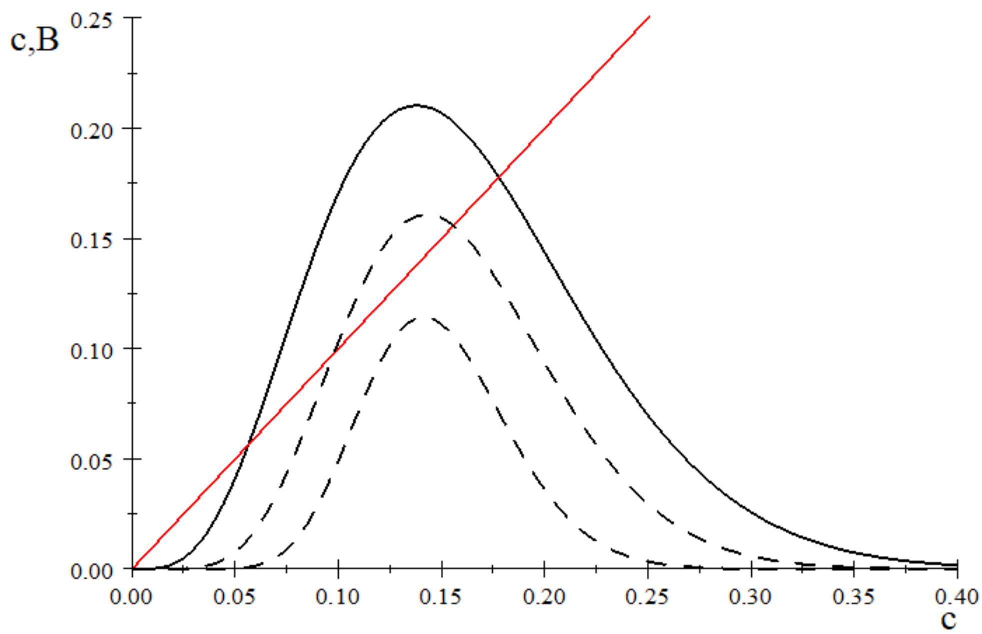

Figure 4: Illustration of fixed point equation (28) for $n = 30$ (solid top curve), $n = 50$ (dashed middle curve), and $n = 100$ (dashed bottom curve).

know that there is a $n^{(1)}$ such that for $n > n^{(1)}$ the probability of a contribution is zero when the remaining contributors are $k_n^2 \geq \alpha n/2$. It follows that at $T = 1$ a players knows that if volunteers at $T = 1$ are not at least $\alpha n/2$, then $k_n^2 \geq \alpha n/2$ and the project fails. The complication is that now a player can receive a positive payoff for any k_n^2 in which $k_n^2 < \alpha n/2$, not just when a specific threshold is reached. There are two possibilities. The first is when $F\left(c_n^{1,2}\right) < \alpha/2$, where $c_n^{1,2}$ is the cutoff at $T = 1$. In this case the probability of at least $k_n^1 \geq \alpha n/2$ contributors at $T = 1$ converges to zero fast, and indeed can be bounded above by

$$H(c_n^{1,2}) = \exp\left(-n \cdot D(\alpha \,\|F(c_n^{1,2}))\right).$$

where $D(\alpha \,\|F(c)) = \exp\left(-n\left(\frac{\alpha}{2}\log\frac{\alpha/2}{F(c)} + (1-\frac{\alpha}{2})\log\frac{1-\alpha/2}{1-F(c)}\right)\right)$ is the Kullback–Leibler divergence. This function of c lies below the 45^o line for n large, just as in Figure (4). The other possibility is that $F\left(c_n^{1,2}\right) \geq \alpha/2$. But this can be ruled out by the following argument. As $n \to \infty$, the expected benefit of contributing for an individual player always converges to zero; but then players with a strictly positive cost near $c = F^{-1}(\alpha/2)$ will not find it optimal to contribute in equilibrium. In any equilibrium sequence we must have $c_n^1 \to 0$, so we are always in the first case in which $F\left(c_n^{1,2}\right) < \alpha/2$ for n sufficiently large.

7 Conclusions

Collective action problems arise when a group's collective goal can only be achieved if at least some fraction of its members engage in a costly action to help the group succeed. We study the equilibrium properties of collective action problems when decisions are taken dynamically over a potentially long horizon, delay is costly, and members have heterogeneous and privately known preferences.

We present two categories of characterization results. The first half of the paper characterizes the properties of dynamic equilibrium, as well as providing efficiency results, for any fixed group size and any fixed participation threshold for group success. The simplest such case is known as the volunteers dilemma, or bystander intervention problem, where exactly one member must undertake the action - to rescue a drowning swimmer, or call 911 to report an accident or ongoing violent crime. This also happens to be the only case that had been analyzed as a dynamic stochastic game with incomplete information (Bliss and Nalebuff, 1984). Our first finding establishes that the dynamic volunteers dilemma is a very special case: except in the limit with arbitrarily large groups, the equilibrium properties of the dynamic version of this game do not extend to the more general, and arguably more realistic, case where group success requires the action of more than one member.

In the volunteers dilemma case where only one member is needed, the group always succeeds as long as at least one member of the group has an action cost that is less than the benefit of success. As soon as group success requires the coordinated action of multiple members, this is no longer true. In this more complicated environment, a member who contemplates taking an action early on faces a real risk that their action will be useless because there will never be a sufficient number of members who decide to activate at later dates. In fact, we show that in such environments, while there will always be a positive probability of success, there is also always a positive probability that the dynamic process of accumulating more activists and getting closer to the goal can fizzle out. In that case, all members who have activated lose out and nobody benefits. The possibility of such failure always exists unless it is common knowledge that every group member would be willing to activate if pivotal. Hence, there are two sources of inefficiency: delay (time is costly); and the positive probability that the goal is not achieved but many members suffer their action cost.

Our second category of results explores the efficiency properties of dynamic equilibrium in large groups. For these results, we allow both the group size as well as the required threshold to grow without bound and obtain a characterization of efficiency in the limit, which depends on the relative rate at which the threshold grows relative to the group size.

If the fraction of members required for the threshold converges to zero very fast as group size increases –at a rate *faster than the cube root of the inverse of the group size*– then there is always a limiting equilibrium where the goal is instantly achieved with probability 1. There is no delay and full efficiency is achieved. The volunteers dilemma is a special case of this.

On the other hand, if the fraction of members required for the threshold converges to zero at a rate *slower than the cube root of the inverse of the group size*, then *in every limiting equilibrium* the probability of group success is 0 and action fizzles out immediately. A special case of this arises if a constant fraction of group size is required. Thus, in both cases, delay costs disappear, as does the deadweight loss incurred when some group members' action costs are wasted. The only efficiency issue in the limit is the probability of group success, which is either 0 or 1.

We also provide some extended results about the robustness of the equilibrium and the possibility of alternative dynamic mechanisms to improve group success. Along these lines, we considered what happens if it is common knowledge that all members have a cost below v. This case is distinctly different, because if the group ever reaches a point with $k = 1$, success is *guaranteed*. We show that this implies if v is sufficiently large (significantly larger than the upper bound of the set of possible costs), then *all equilibria lead to eventual success*. Second, we consider the possibility that there is aggregate uncertainty about the distribution of costs. The characterization of the dynamic evolution of equilibrium cutpoints takes a similar form as the basic model, but with the added twist that the group learns over time about the distribution of costs. This learning can lead members to become more optimistic or more pessimistic about the ultimate chances of group success, depending on the early realizations of some members' activation decisions. Third we show how non-stationarities in the value of group success can be incorporated into the model. These extensions to aggregate uncertainty and non-stationarities raise interesting questions about bandwagon effects and their effect on the likelihood of group success. Fourth, we explore the possibility that group leadership exists at least to the degree that they can commit to a finite deadline, essentially using it as a threat to encourage early participation and discourage procrastination. However, it turns out that committing to such deadlines is actually counter-productive. Specifically, we show that for any deadline T there exists a group size $n(T)$ such that all groups larger than $n(T)$ *completely fail*, in the sense that the only equilibrium involves no member ever participating, even those with arbitrarily low costs.

The finer details of the equilibrium dynamics warrant further study. In collective action problems associated with protest movements, petition drives, and similar environments that unfold over time, one suspects that the equilibrium will reflect bandwagon effects. That is, if the protest movements or petition drives catch on quickly and exhibit heavy participation from the outset, one expects that this early activity will snowball and encourage others to join in because the prospects of success are higher. On the flip side, if there are only a handful of demonstrators or visible activists in the early stages, then the group members who were sitting on the sidelines might decide to just forget about it and the movement would fizzle out. We do not have explicit results about this in the form of comparative statics properties of equilibrium, but this would seem to be a useful direction to pursue.

8 Appendix

8.1 Proof Lemma 1

We proceed in two steps.

Step 1. We first prove that the expected continuation value of a player who does not volunteer is convex in c and admits right and left derivatives, respectively denoted $\partial^r \left[V^1\right]^- (c, h_t^1)/\partial c$ and $\partial^l \left[V^1\right]^- (c, h_t^1)/\partial c$, with $\partial^d \left[V^1\right]^- (c, h_t^1)/\partial c > -e^{-\gamma\Delta}$ for $d = r, l$. Consider any history h_t^1 in which only one volunteer is missing for success, and denote h_{t+j}^1 any history following h_t^1 in which no there is volunteer from t to $t+j$. Let $\beta(h_t^1)$ be the probability that there is at least 1 volunteer at history h_t^1. Define $W^{1,\lambda}(c, h_t^1)$ to be the expected value to a player of type c at h_t^1 who does not volunteer at t, but instead volunteers after $\lambda \in [1, \infty)$ periods (if there is not a volunteer before):

$$\begin{aligned} W^{1,\lambda}(c, h_t^1) &= \sum\nolimits_{\tau=0}^{\lambda-1} e^{-\gamma\Delta\tau} \cdot \left[\prod\nolimits_{j=0}^{\tau} \left[1 - \beta(h_{t+j-1}^1)\right]\right] \beta(h_{t+\tau}^1) v \\ &\quad + e^{-\gamma\Delta\cdot\lambda} \cdot \prod\nolimits_{j=0}^{\lambda} \left[1 - \beta(h_{t+j-1}^1)\right] \cdot (v - c) \end{aligned} \tag{29}$$

where we define by convention $\beta(h_{t-1}^1) = 0$. Note that all these expressions are linear in c and $\partial W^{1,\lambda}(c, h_t^1)/\partial c > -e^{-\gamma\Delta} > -1$. The value for a player who does not volunteer at h_t^1 is:

$$\left[V^1\right]^- (c, h_t^1) = \max_{\lambda} W^{1,\lambda}(c, h_t^1)$$

which is convex in c and so admits right and left derivatives with $\partial^d \left[V^1\right]^- (c, h_t^1)/\partial c > -e^{-\gamma\Delta}$ for $d = r, l$.

Step 2. Note that at h_t^1, the expected utility of a type c who volunteers at t is $\left[V^1\right]^+ (c, h_t^1) = v - c$. Suppose now a type $c(h_t^1)$ is indifferent between volunteering or not, so:

$$v - c(h_t^1) = \left[V^1\right]^- (c(h_t^1), h_t^1) \tag{30}$$

Consider a type $c' > c(h_t^1)$ with $\Delta c = c' - c(h_t^1)$. We have:

$$\begin{aligned} v - c' &= v - c(h_t^1) - \Delta c = \left[V^1\right]^- (c(h_t^1), h_t^1) - \Delta c \\ &< \left[V^1\right]^- (c(h_t^1), h_t^1) - e^{-\gamma\Delta}\Delta c < \left[V^1\right]^- (c', h_t^1) \end{aligned}$$

where in the second we use (30), and in the last inequality we use the convexity of $\left[V^1\right]^- (c', h_t^1)$. The proof that $c' < c(h_t^1)$ implies $v - c > \left[V^1\right]^- (c', h_t^1)$ is analogous. ■

8.2 Proof of Lemma 2

First note that $c(h_t) \in [l_{h_t}, v)$ and suppose to the contrary that $c(h_t) = l_{h_t}$ in the PBE, i.e., no member will activate in period t. This implies $V^-(c, h_t) > V^+(c, h_t)$ for all $c(h_t) \in (l_{h_t}, v)$. But it is easy to show that if no member will activate in period t, then it must be that no member will activate in period $t+1$ as well. To see this, notice that if a member with cost $c \in [l_{h_t}, v)$ were using a strategy to activate in period $t+1$ and the member knows that no member is activating in period t, then their payoff in the continuation game is $V^-(c, h_t) = e^{-\gamma\Delta}(v - c) < v - c = V^+(c, h_t)$, a contradiction. It follows that if no player volunteers at t, $V^-(c, h_t) = 0 < v - c = V^+(c, h_t)$, implying again a contradiction. We conclude that at t the probability a player volunteers is strictly positive. To prove that $\lim_{t\to\infty} c(h_t) = v$, suppose to the contrary that $\lim_{t\to\infty} c(h_t) = \overline{c} < v$. Then $\lim_{t\to\infty} V^-(\overline{c}, h_t) = 0 < V^+(\overline{c}, h_t) = v - \overline{c} > 0$, a contradiction. ■

8.3 Proof of Lemma 3

We proceed in two steps.

Step 1. We first prove that the cutpoints are declining in n so: $c_t(n) < c_t(n-1)$ for all $t > 0$. To see this note that $\left(1 - e^{-\gamma\Delta}\right) x / \left[1 - e^{-\gamma\Delta} \cdot x\right]$ is a strictly increasing function of x if, as always verified in our environment, $e^{-\gamma\Delta} \cdot x < 1$. Since $1 - F(c)/[1 - F(l)] < 1$ for $c > l$, it follows that the function:

$$\lambda_n(c; l, \gamma, \Delta) = \frac{\left(1 - e^{-\gamma\Delta}\right)\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}{1 - e^{-\gamma\Delta} \cdot \left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}} \tag{31}$$

is strictly decreasing in n for any c, l, γ, Δ. It can also be verified that $\lambda_n(c; l, \gamma, \Delta)$ is strictly decreasing in c for all n, l, γ, Δ. From (31), it follows that the fixed point $c_{n,1}^1$ is decreasing in n for any l, γ, Δ. To see this, note that:

$$0 = \lambda_n(c_1(n); l, \gamma, \Delta) - c_1(n) < \lambda_{n-1}(c_1(n); l, \gamma, \Delta) - c_1(n)$$

where the equality follows by the definition of $c_1(n)$, and the inequality follows by the monotonicity of $\lambda_n(c; l, \gamma, \Delta)$ in n. Since $\lambda_{n-1}(c; l, \gamma, \Delta) - c$ is strictly decreasing in c, it follows that we must have $c_1(n-1) > c_1(n)$, else it would be $\lambda_{n-1}(c_1(n-1); l, \gamma, \Delta) - c_1(n-1) > 0$, a contradiction. Assume the induction hypothesis that $c_j(n)$ is decreasing in n for all $j \leq t$. We prove the same is true for $j = t+1$. To see this note that an increase in n shifts the function $\left(\frac{1-F(c)}{1-F(c_t(n))}\right)^{n-1}$ downward for any c since it increases the exponent while reducing $F(c_t(n))$, thus increasing the denominator. It follows that $\lambda_n(c; c_t(n), \gamma, \Delta)$ shifts downward for any c after an increase in n, implying as above that:

$$0 = \lambda_n(c_{t+1}(n); c_t(n), \gamma, \Delta) - c_{t+1}(n) < \lambda_{n-1}(c_{t+1}(n); c_t(n), \gamma, \Delta) - c_{t+1}(n)$$

thus implying that $c_{t+1}(n-1) > c_{t+1}(n)$. This proves the first part of the lemma.

Step 2. We now prove that $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. The probability of success at or before period t, $\Phi_t(n)$, can be written as:

$$\Phi_t(n) = 1 - (1 - F(c_t(n)))^{n-1} = 1 - \prod_{j=1}^{t} \left(\frac{1 - F(c_j(n))}{1 - F(c_{j-1}(n))}\right)^{n-1}$$

From the cutpoint condition in equation (4) we have:

$$c_j(n) = \left[\frac{\left(1 - e^{-\gamma\Delta}\right)\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-m}}{1 - e^{-\gamma\Delta} \cdot \left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-m}}\right] v$$

Since the right hand side is increasing in $\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-m}$, it follows that $\left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-1}$ is decreasing in n for any j, since $c_j(n)$ is decreasing in n. We conclude that an increase in n induces an increase in $1 - \prod_{j=1}^{t} \left(\frac{1-F(c_j(n))}{1-F(c_{j-1}(n))}\right)^{n-1}$, and thus in $\Phi_t(n)$. It follows that $\Phi_t(n) \geq \Phi_t(n-1)$ for all t and $\Phi_t(n-1)$ first order stochastically dominates $\Phi_t(n)$. ■

8.4 Proof of Theorem 1

If $c \leq c_1(n)$, then a type c has a payoff of $v - c$ irrespective of the total number of players. If $c \in [c_1(n), c_1(n-1)]$, then with $n-1$ players the payoff of a type c is $v - c$ and with n players the payoff of a type c is not lower than $v - c$ by revealed preferences, strictly in $(c_1(n), c_1(n-1)]$. We now prove the result by induction, using these findings as first step. Assume we have proven that for all types $c \leq c_t(n-1)$ for $t \leq j$ a player of type c with n players has utility $EU_n(c)$, weakly higher than the utility of a type c with $n-1$ players $EU_{n-1}(c)$. We have just proven this result for $j = 1$.

Consider first a type $c \in [c_j(n-1), \min\{c_{j+1}(n), c_{j+1}(n-1)\}]$, if not empty. When there are $n-1$ players, the payoff of a type c is:

$$EU_{n-1}(c) = v\Phi_j(n-1)e^{-\gamma\Delta(t-1)} + [1 - \Phi_j(n-1)]\, e^{-\gamma\Delta j}\,(v - c)\,. \tag{32}$$

A type c with n players instead receives:

$$EU_n(c) = v\Phi_j(n)e^{-\gamma\Delta(t-1)} + [1 - \Phi_j(n)]\, e^{-\gamma\Delta j} \cdot (v - c) > V_{n-1}(c).$$

where the last inequality follows from the fact that $\Phi_t(n-1)$ strictly first order stochastically dominates $\Phi_t(\Delta, n)$ and $v > v - c$.

Alternatively, it could be that $[c_j(n-1), \min\{c_{j+1}(n), c_{j+1}(n-1)\}]$ is empty. In that case, consider $c \in [\min\{c_{j+1}(n), c_{j+1}(n-1)\}, c_{j+1}(n-1)]$, which must be nonempty. In this case the payoff of a type c with $n-1$ players is again (32), which can be rewritten as:

$$EU_{n-1}(c) = v\Phi_j(n-1)e^{-\gamma\Delta(t-1)} + \sum_{t=j}^{\infty} [\Phi_{t+1}(n-1) - \Phi_t(n-1)]\, e^{-\gamma\Delta j} \cdot (v - c)$$

The payoff with n of a player with type c instead is:

$$EU_n(c) = v\Phi_j(n)e^{-\gamma\Delta(t-1)} + \sum_{t=j}^{\infty} [\Phi_{t+1}(n) - \Phi_t(n)]\, e^{-\gamma\Delta j} \cdot V_{n,j}(c)$$

where $V_{n,j}(c)$ is the expected continuation value function for a type c when there are n players and j contributors; and where we have $V_{n,j}(c) \geq (v - c)$ by revealed preferences, since $c > c_{j+1}(n)$. Once again we have that $V_{n-1}(c) < V_n(c)$. We have therefore proven that for all $c \leq c_{j+1}(n)$, we have $V_{n-1}(c) < V_n(c)$. It follows that for all types $c < \lim_j c_{j+1}(n) = v$, we have $EU_{n-1}(c) < EU_n(c)$, which proves the result. ■

8.5 Proof of Lemma 4

We proceed again in two steps.

Step 1. We first prove that the cutpoints are declining in Δ so: $c_t(\Delta) < c_t(\Delta')$ for all t and $\Delta > \Delta'$. To see this note that $\frac{(1-e^{-\gamma\Delta})\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}{1-e^{-\gamma\Delta}\cdot\left(\frac{1-F(c)}{1-F(l)}\right)^{n-1}}$ is a strictly increasing function of Δ. It follows that $\lambda_n(c; l, \gamma, \Delta)$, as defined in (31), is strictly increasing in Δ for any c, l, γ, n. By the same argument as in Lemma 3, it follows from (31) that the fixed point $c_1(\Delta)$ is increasing in Δ for any l, γ, n. Assume the induction hypothesis that $c_j(\Delta)$ is increasing in Δ for all $j \leq t$. We now prove the same is true for $j = t+1$. To see this note that an increase in Δ certainly increases $\left(\frac{1-F(c)}{1-F(c_t(\Delta))}\right)^{n-1}$ for any given c, since it increases $F(c_t(\Delta))$. It follows that $\lambda_n(c; c_t(\Delta), \gamma, \Delta)$ shifts upward after an increase in Δ, implying that $c_{t+1}(\Delta)$ increases. This proves the first part of the lemma.

Step 2. We now prove that $\Phi_t(\Delta)$ first order stochastically dominates $\Phi_t(\Delta')$ for $\Delta < \Delta'$. As in the proof of Lemma 3, we define:

$$\Phi_t(\Delta) = 1 - \left(\frac{1 - F(c_t(\Delta))}{1 - F(l)}\right)^{n-1}$$

Since $\left(\frac{1-F(c_j(\Delta))}{1-F(l)}\right)^{n-1}$ is decreasing in Δ for all j, then $\Phi_t(\Delta') \leq \Phi_t(\Delta)$ for all t and $\Delta < \Delta'$. ■

8.6 Proof of Proposition 1

For any $\varepsilon > 0$, the probability of the event E in which no player has a type c lower than ε is $[1 - F(\varepsilon)]^n > 0$. In this event, no success can occur until we reach a period t in which the cutpoint is strictly larger than ε. Let t_ε be the minimal t such that $c_{t_\varepsilon} \geq \varepsilon$. For Δ small, we can assume without loss of generality that $c_{t_\varepsilon - 1} > \frac{\varepsilon}{2}$. To see this note that if $c_{t_\varepsilon - 1} \leq \frac{\varepsilon}{2}$, then (4) implies that for Δ sufficiently small, c_{t_ε} is arbitrarily close to $\frac{\varepsilon}{2}$ as well: so $c_{t_\varepsilon} < \varepsilon$, a contradiction. The utility of a player in event E is not larger than $(v - \frac{\varepsilon}{2})$ since in period $t_\varepsilon - 1$ no player can obtain a payoff larger than the lowest type. But then $\Delta \to 0$, the utility of a player is at most $(1 - [1 - F(\varepsilon)]^n)\, v + [1 - F(\varepsilon)]^n\, (v - \varepsilon/2) = v - [1 - F(\varepsilon)]^n \frac{\varepsilon}{2} < v$. The result is proven if we let $\delta = [1 - F(\varepsilon)]^n \frac{\varepsilon}{2}$. ■

8.7 Proof of Lemma 5

We proceed in three steps.

Step 1. Lemma 1 already established that there is a unique PBE when $k = 1$, which is in cutoff strategies. In that equilibrium, for any history at which $k = 1$, the value for an active player who volunteers is $\left[V^1\right]^+ (c, h_t^1) = v - c$, which is linear in c with $\partial \left[V^1\right]^+ (c, h_t^1)/\partial c = -1$. The proof of Lemma 1 also established that $\left[V^1\right]^- (c, h_t^1)$ is piecewise linear, convex and hence admits right and left derivatives $\partial^r \left[V^1\right]^- (c, h_t^1)/\partial c$ and $\partial^l \left[V^1\right]^- (c, h_t^1)/\partial c$, both bounded below by $-e^{-\gamma\Delta} > -1$.

Step 2. At any history h_t^k, the continuation value of a player who has *previously* committed (i.e. $Q^k(l_{h_t^k})$) does not depend on his/her own type c; it depends only of the behavior of the other players. It follows that the value of an active player at h_t^k, with cost c, who volunteers at h_t^k, i.e. $\left[V^k\right]^+ (c, h_t^k)$, is linear in c with $\partial \left[V^k\right]^+ (c, h_t^k)/\partial c = -1$. We now proceed by induction. Assume that, for all $\kappa = 1, ..., k-1$, $[V^\kappa]^- (c, h_t^\kappa)$ is convex in c, with $\partial^d [V^\kappa]^- (c, h_t^\kappa)/\partial c \geq -e^{-\gamma\Delta}$ for $d = l, r$. (Step 1 established this property for $\kappa = 1$.) We need to prove that the same is true for $\left[V^k\right]^- (c, h_t^k)$.

To see this, first observe that the value function at h_t^κ is:

$$[V^\kappa]\,(c, h_t^\kappa) = \max\left\{[V^\kappa]^+ (c, h_t^\kappa), [V^\kappa]^- (c, h_t^\kappa)\right\}$$

for any $\kappa < k$. Since $[V^\kappa]^+ (c, h_t^\kappa)$ is linear in c and $[V^\kappa]^- (c, h_t^\kappa)$ is convex in c , then $[V^\kappa]\,(c, h_t^\kappa)$ is convex in c for all $\kappa < k$.

Define $\vartheta(h_t^k)$ to be the probability of having at least one volunteer at h_t^k. Define $\Phi^k(c, h_t^k)$ to be the expected utility of a type c player at history h_t^k conditioning on at least one volunteer at history h_t^k. Because of the properties of $[V^\kappa](c, h_t^\kappa)$ for $\kappa < k$ proven above, $\Phi^k(c, h_t^k)$ is convex in c and $\partial^d\Phi^k(c, h_t^k)/\partial c \geq -e^{-\gamma\Delta}$ for any $d = l, r$. Finally, define $W^{k,\lambda}(c, h_t^k)$ to be the value of a player of type c at h_t^k who does not volunteer at t, but volunteers instead after at most $\lambda \in [1, \infty)$ periods in which there is no other volunteer, if there is no volunteer before. We can write:

$$\begin{aligned} W^{k,\lambda}(c, h_t^k) &= \sum\nolimits_{\tau=0}^{\lambda-1} e^{-\gamma\Delta\tau} \cdot \left[\prod\nolimits_{j=0}^{\tau} \left[1 - \vartheta(h_{t+j-1}^k)\right]\right] \cdot \vartheta(h_{t+\tau}^k) \cdot \Phi^k(c, h_{t+\tau}^k) \\ &\quad + e^{-\gamma\Delta\cdot\lambda} \cdot \left[\prod\nolimits_{j=0}^{\lambda} \left[1 - \vartheta(h_{t+j-1}^k)\right])\right] \cdot \left[V^k\right]^+ (c, h_{t+\lambda}^k) \end{aligned} \tag{33}$$

where $\vartheta(h_{-1}^k) = 0$ by convention. Since $\Phi^k(c, h_{t+\tau}^k)$ and $\left[V^k\right]^+(c, h_t^k)$ are convex $\forall\tau = 1, ..., \lambda - 1$, $W^{k,\lambda}(c, h_t^1)$ is convex. And by the same argument as before, this also implies that $\partial^d W^{k,\lambda}(c, h_t^k)/\partial c \geq -e^{-\gamma\Delta}$ for any $d = l, r$.

Finally, note that

$$\left[V^k\right]^- (c, h_t^k) = \max_\lambda W^{k,\lambda}(c, h_t^k)$$

It follows that $\left[V^k\right]^-(c, h_t^k)$ is convex in c and $\partial^d \left[V^k\right]^-(c, h_t^k)/\partial c \geq -e^{-\gamma\Delta}$ for $d = l, r$.

Step 3. Assume now a type $c(h_t^k)$ is indifferent between volunteering or not at history h_t^k. That is:

$$\left[V^k\right]^+ (c(h_t^k), h_t^k) = \left[V^k\right]^- (c(h_t^k), h_t^k) \tag{34}$$

Consider a type $c' > c(h_t^k)$ with $\Delta c = c' - c(h_t^k)$. We have:

$$\begin{aligned} \left[V^k\right]^+ (c', h_t^k) &= \left[V^k\right]^+ (c(h_t^k), h_t^k) - \Delta c = \left[V^k\right]^- (c(h_t^k), h_t^k) - \Delta c \\ &< \left[V^k\right]^- (c(h_t^k), h_t^k) - e^{-\gamma\Delta}\Delta c < \left[V^k\right]^- (c', h_t^k) \end{aligned}$$

where in the first equality we use the linearity of $\left[V^k\right]^+(c', h_t^k)$, in the second equality we use (34), in the third (inequality) we use the induction hypothesis, and finally in the last inequality we use the convexity of $\left[V^k\right]^-(c', h_t^k)$. The proof that $c' < c(h_t^k)$ implies $\left[V^k\right]^+(c', h_t^k) > \left[V^k\right]^-(c', h_t^k)$ is analogous. ■

8.8 Proof of Theorem 3

We proceed in three steps.

Step 1. For any history, h_t^k with k missing volunteers and lower bound $l_{h_t^k} = l$ define the set of possible equilibrium cutpoints as:

$$Z(l; w) = \left\{ c \geq l \;\middle|\; \begin{array}{c} c = e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\begin{array}{c} \left(Q^{k-j-1}(c) - V^{k-j}(c, c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array} \right] \\ \text{or } c = l \text{ if } e^{-\gamma\Delta} Q^{k-1}(l) \leq l, \\ \text{for some: } \{Q^{k-j-1}(c), V^{k-j}(c, c)\}_{j=1}^{k-1} \in \{\mathcal{V}^{k-j}(c)\}_{j=1}^{k-1}, \\ \text{and } V^k(c, c) = w \end{array} \right\} \tag{35}$$

where w is the continuation value following the history if no member volunteers at stage t. The set of all possible future equilibrium value functions in the definition of $Z(l;w)$, $\{\mathcal{V}^{k-j}(c)\}_{j=1}^{k-1}$, is defined by the induction hypothesis for all $j \in 1,2,...,k-1$, i.e. when at least one volunteer activates in stage t; and $\{Q^{k-j-1}(c), V^{k-j}(c,c)\}_{j=1}^{k-1}$ is a selection from $\{\mathcal{V}^{k-j}(c)\}_{j=1}^{k-1}$: i.e., $\{Q^{k-j-1}(c), V^{k-j}(c,c)\}_{j=1}^{k-1}$ is the collection of future equilibrium value functions associated with one specific PBE. Note that we have proven in Section 3 that $V^1(c,c)$ is a continuous function of c, so a fortiori $\mathcal{V}^1(c)$ is upper-hemicontinuous in c. We now assume as induction hypothesis that $\mathcal{V}^{k-j}(c)$ is upper-hemicontinuous for all $j \in [1,k-1]$.

For any possible equilibrium cutoff, $c_w \in Z(l;w)$, and associated set of future equilibrium value functions, $\{Q^{k-j-1}(c_w), V^{k-j}(c_w,c_w)\}_{j=1}^{k-1} \in \{\mathcal{V}^{k-j}(c_w)\}_{j=1}^{k-1}$ one obtains the corresponding value functions $Q_w^k(l), \left[V_w^k\right]^+(c,l)$, $\left[V_w^k\right]^-(c,l)$ and $V_w^k(c,l) = \max\{\left[V_w^k\right]^+(c,l), \left[V_w^k\right]^-(c,l)\}$. These functions directly depend on both c_w and w, since w is the expected continuation value in case of no volunteers. Note that $V_w^k(c,l) = \max\{\left[V_w^k\right]^+(c,l), \left[V_w^k\right]^-(c,l)\}$ is continuous in c since $\left[V_w^k\right]^+(c,l)$ and $\left[V_w^k\right]^-(c,l)$ are both continuous in c. Let $\mathcal{E}^k(l;w)$ be the set of possible equilibrium values for an uncommitted player of type l when the lower bound on types is l and k volunteers are missing; and denote the convex hull of $\mathcal{E}^k(l;w)$ by $\Delta\mathcal{E}^k(l;w)$. Note that the set $\Delta\mathcal{E}^k(l;w)$ corresponds to the equilibrium values if the expected continuation values are PBE when at least one volunteer activates in period t, and equal to w is the value if there is no contribution. Values in the interior of the convex hull correspond to situations in which the public randomization device is used to mix between equilibria in the subgame, for example mixing between the equilibria generating $V_w^k(l,l)$ and $\widetilde{V}_w^k(l,l)$, where both $V_w^k(c,l)$ and $\widetilde{V}_w^k(c,l)$ are equilibrium continuation values functions in $\mathcal{V}^k(c_w)$, constructed as described above.

Step 2. Define now as initial steps of a sequence, $Z_0(l) = Z(l;0)$, the set of equilibrium cutpoints if the expected continuation value in case of no contributions is 0 (i.e., the game were to be immediately terminated if there are no contributions), and $\Delta\mathcal{E}_0^k(l) = \Delta\mathcal{E}^k(l;0)$, the convex hull of the corresponding set of value functions. For each $\tau = 1,2,...$, given $\Delta\mathcal{E}_{\tau-1}^k(l)$, recursively define $Z_1(l), Z_2(l), ...$ similarly, that is:

$$Z_\tau(l) = \left\{ c \geq l \;\middle|\; \begin{array}{c} c = e^{-\gamma\Delta}\sum_{j=0}^{k-1}\left[\begin{array}{c} \left(Q^{k-j-1}(c) - V^{k-j}(c,c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array}\right] \\ \text{or } c = l \text{ if } e^{-\gamma\Delta}Q^{k-1}(l) \leq l \\ \text{for some: } Q^{k-j-1}(c), V^{k-j}(c,c) \in \mathcal{V}^{k-j}(c)\ \forall j\ j \in [1,k-1] \\ \text{and } V^k(c,c) \in \Delta\mathcal{E}_{\tau-1}^k(c) \end{array} \right\} \tag{36}$$

In other words, the set $Z_\tau(l)$ is the set of cutpoints that can be an equilibrium at a history with k missing volunteers if the game were to be terminated after τ periods of no additional volunteers. We call these the set of possible cutpoints for the *τ-truncated game,* and it defines a sequence of sets of possible cutpoints, $\{Z_\tau(l)\}_{\tau=1}^\infty$. If $e^{-\gamma\Delta}Q^{k-1}(l) \leq l$, then $Z_\tau(l)$ is obviously not empty. If $e^{-\gamma\Delta}Q^{k-1}(l) > l$, note that the correspondence in c defined by

$$\varphi_l^\tau(c) = \left\{ \begin{array}{c} x \in [l,1] \left| x = e^{-\gamma\Delta}\sum_{j=0}^{k-1}\left[\begin{array}{c} \left(Q^{k-j-1}(c) - V^{k-j}(c,c)\right) \\ \cdot B\left(j, n-1-m+k, \frac{F(c)-F(l)}{1-F(l)}\right) \end{array}\right] \right. \\ \text{for some } [Q^{k-j-1}(c), V^{k-j}(c,c)] \in \mathcal{V}^{k-j}(c)\ \forall j \in [1,k-1] \\ \text{and } V^k(c,c) \in \Delta\mathcal{E}_{\tau-1}^k(c) \end{array} \right\}$$

is non empty, convex- and closed-valued since $Q^{k-j-1}(c), V^{k-j}(c,c) \in \mathcal{V}^{k-j}(c)$ and $V^k(c,c) \in \Delta\mathcal{E}^k_{\tau-1}(c)$. Moreover, since $\mathcal{V}^{k-j}(c)$ and $\Delta\mathcal{E}^k_{t-1}(c)$ are upper-hemicontinuous in c, $\varphi^\tau_l(c)$ is upper-hemicontinuous in c as well. It follows that $\varphi^\tau_l(c)$ is closed valued and upper-hemicontinuous in c and hence has a closed graph. We conclude that $\varphi^\tau_l(c)$ is non-empty, convex-valued and has closed graph in c, so by the Kakutani fixed-point theorem implies it has a fixed point. This implies $Z_\tau(l)$ is non empty, since any fixed point of $\varphi^\tau_l(c)$ is an element of $Z_\tau(l)$. For each $\widetilde{c} \in Z_\tau(l)$, we can construct the corresponding value functions $Q^k_\tau(l)$ and $V^k_\tau(c,l)$, which are continuous in c. Define $\Delta\mathcal{E}^k_\tau(l)$ to be the convex hull of the set of continuation values for a type l when the lower bound is l. The set $\Delta\mathcal{E}^k_\tau(l)$ is non empty, convex and closed valued and upper-hemicontinuous in l. To verify the last property, note that for any sequence $\{l_\iota\} \to l$, we can select a corresponding sequence of $c^k_\tau(l_\iota) \in Z_\tau(l_\iota)$ and define the corresponding values for uncommitted players, $V^k_\tau(c,l_\iota)$. Let $c^k_\tau(l) = \lim_{\iota\to\infty} c^k_\tau(l_\iota)$, since $c^k_\tau(l_\iota) \in Z_\tau(l_\iota)$ for all ι, then we must have at least a subsequence in which either the first or the second line of (36) is true for all ι: this implies that $c^k_\tau(l)$ satisfies the first or the second line of (36) as well, so $c^k_\tau(l) \in Z_\tau(l)$. Note that $\lim_{\iota\to\infty} V^k_\tau(c,l_\iota) = V^k_\tau(c,l)$ and $V^k_\tau(c,l)$ is an equilibrium value function since $c^k_\tau(l) \in Z_\tau(l)$. So, for any sequence $\{l_\iota\} \to l$, there is a selection $V^k_\tau(l_\iota,l_\iota) \in \Delta\mathcal{E}^k_\tau(l_\iota)$ with $V^k_\tau(l_\iota,l_\iota) \to V^k_\tau(l,l)$, such that $V^k_\tau(l,l) \in \Delta\mathcal{E}^k_\tau(l)$.

Step 3. Consider now a sequence of cutoffs $c^k_\tau(l) \in Z_\tau(l)$ as $\tau \to \infty$, and the associated value functions $Q^k_\tau(l)$ and $V^k_\tau(c,l)$. We can define $c^k(l) = \lim_{\tau\to\infty} c^k_\tau(l)$ and the associated (limiting) value functions $Q^k(l)$ and $V^k(c,l)$. We claim that this is a PBE. Assume this is not true. Then there is a deviation for a player i that yields $\overline{V}^k_i(c,l)$ such that $\overline{V}^k_i(c,l) - V^k(c,l) > 2\varepsilon$ for some $\varepsilon > 0$. We now make two observations. First, let $\overline{V}^k_{i,\tau}(c,l)$ be the value of the strategies used in $\overline{V}^k_i(c,l)$ in the τ-truncated game, i.e. under the constraint that the game is terminated if there are τ periods without volunteers. Since utilities are bounded and $\Delta, \gamma > 0$, the truncated game is continuous at infinity (as defined in Fudenberg and Levine, 1983). We must therefore have $\overline{V}^k_{i,\tau}(c,l) \geq \overline{V}^k_i(c,l) - \varepsilon/2$ for τ sufficiently large. Moreover, by construction, $V^k_\tau(c,l) \leq V^k(c,l) + \varepsilon/2$ for τ sufficiently large. It follows that there exists a τ^* such that for $\tau > \tau^*$:

$$\overline{V}^k_{i,\tau}(c,l) - V^k_\tau(c,l) \geq \varepsilon$$

But this is in contradiction with the fact that $V^k_\tau(c,l)$ is the equilibrium value function of a PBE in the τ-truncated game.

Step 4. We conclude the induction step by proving that the set of equilibrium values $\mathcal{V}^k(l)$ is nonempty, closed, convex-valued and upper-hemicontinuous in l. We showed above that $Z(l)$ is non empty. We now prove that $Z(l)$ is upper hemicontinuous, which immediately implies the desired result. Consider a sequence $\{l_\iota\} \to l$ and the associated sequence $c^k(l_\iota) \in Z(l_\iota)$. We need to prove that if $c^k(l_\iota) \to \lim_{\iota\to\infty} c^k(l_\iota)$, then $\lim_{\iota\to\infty} c^k(l_\iota) \in Z(l)$. To show this, define $c^k(l_\iota) \in \arg\min_{c^k \in Z(l_\iota)} \left|c^k - \lim_{j\to\infty} c^k(l_j)\right|$ and assume by contradiction that $\left|c^k(l) - \lim_{j\to\infty} c^k(l_j)\right| > \varepsilon$ for some $\varepsilon > 0$. We can write:

$$\begin{aligned}\left|c^k(l) - \lim_{\iota\to\infty} c^k(l_\iota)\right| \quad &\leq \quad \left|c^k(l) - c^k_\tau(l)\right| + \left|c^k_\tau(l) - c^k_\tau(l_\iota)\right| \\ &\quad + \left|c^k_\tau(l_\iota) - c^k(l_\iota)\right| + \left|c^k(l_\iota) - \lim_{j\to\infty} c^k(l_j)\right|\end{aligned}$$

where $\left\{c^k_\tau(l)\right\}$ is a sequence of equilibrium cutpoints in the truncated game such that $c^k_\tau(l) \to c^k(l)$. Note that by definition of $c^k_\tau(l)$, there is a τ^* such that for $\tau > \tau^*$, $\left|c^k(l) - c^k_\tau(l)\right| < \varepsilon/4$ and

$\left|c_\tau^k(l_\iota) - c^k(l_\iota)\right| < \varepsilon/4$. Similarly, by definition of a limit, there is a ι^* such that for $\iota > \iota^*$, $\left|c^k(l_\iota) - \lim_{j\to\infty} c^k(l_j)\right| < \varepsilon/4$. Finally, note that for a given τ, $c_\tau^k(l)$ is upper-hemicontinuous so it admits a selection such that $\lim_{j\to\infty} c_\tau^k(l_j) \in Z_\tau(l)$, implying that $\left|c_\tau^k(l) - c_\tau^k(l_\iota)\right| < \varepsilon/4$ for $\iota > \iota^*$ and some $c_\tau^k(l) \in Z_\tau(l)$ (if ι^* is chosen sufficiently large). We conclude that $\left|c^k(l) - \lim_{j\to\infty} c^k(l_j)\right| < \varepsilon$ for ι sufficiently large, a contradiction. ■

8.9 Proof of Lemma 6

We proceed in three steps.

Step 1. We first prove that if $m_n \prec n^{2/3}$ then $\frac{F[vB(m_n-1,n-1,\alpha_n)]}{\alpha_n} \to \infty$, where $\alpha_n = m_n/n$. To establish this property, note that we can write:

$$B\left(m_n-1,n-1,\alpha_n\right) = \binom{n-1}{m_n-1}\frac{\left[(\alpha_n)^{\alpha_n}\left(1-\alpha_n\right)^{(1-\alpha_n)}\right]^n}{\alpha_n} \simeq \frac{1}{\sqrt{2\pi\alpha_n(1-\alpha_n)n}}$$

by Stirling's formula. Furthermore, F is approximately uniform in the neighborhood of 0, and $vB\left(m_n-1,n-1,\alpha_n\right) \to 0$, so:

$$\frac{F\left[vB\left(m_n-1,n-1,\alpha_n\right)\right]}{\alpha_n} \simeq \frac{f(0)\cdot vB\left(m_n-1,n-1,\alpha_n\right)}{\alpha_n} \simeq \frac{vf(0)}{\sqrt{2\pi\left(\frac{m_n}{n^{2/3}}\right)^3\left(1-\frac{m_n}{n}\right)}}$$

which diverges to infinity if $m_n \prec n^{2/3}$. Note that this also implies $\frac{F[vB(m_n-1,n-1,L\alpha_n)]}{L\alpha_n} \to \infty$ for any $L > 1$.

Step 2. Recall from Theorem 4 that in every equilibrium the cutpoint at the initial period $t = 1$ is strictly positive, and hence is given by (15) evaluated at $k = m_n$ and $l = 0$:

$$c_{1,n}^{m_n}(0) = e^{-\gamma\Delta}\sum\nolimits_{j=0}^{m_n-1} B\left(j,n-1,F\left(c_n^{m_n}(0)\right)\right)\left[\begin{array}{c} Q^{m_n-j-1}\left(c_n^{m_n}(0)\right) \\ -\left[V^{m_n-j}\right]^+\left(c_n^{m_n}(0),c_n^{m_n}(0)\right)\end{array}\right]. \quad (37)$$

The maximal fixed-point consistent with (37) can be bounded below by $\overline{c}_n^{m_n}(0)$ defined as follows:

$$\overline{c}_n^{m_n}(0) = \max_{c\in[0,1]}\left[c|c \le e^{-\gamma\Delta}\sum\nolimits_{j=0}^{m_n-1} B\left(j,n-1,F\left(c\right)\right)\left[Q^{m_n-j-1}\left(c\right) - \left[V^{m_n-j}\right]^+\left(c,c\right)\right]\right] \quad (38)$$

For any arbitrary constant $L > 1$, define $\widehat{c}_n^L$ by $F\left(\widehat{c}_n^L\right) = L\frac{m_n}{n}$ for n large enough so that $L\frac{m_n}{n} < 1$. That is, given L, $\widehat{c}_n^L$ is a hypothetical cutpoint with the property that the expected fraction of types lower than or equal to $\widehat{c}_n^L$ is greater than the required threshold fraction by a factor of $L > 1$.

We next show that if we choose a sufficiently large (but still finite) value of L then there will exist a critical group size n_L such that for $n > n_L$:

$$\begin{aligned}\Psi_{m_n,n}\left(\widehat{c}_n^L\right) &\equiv e^{-\gamma\Delta}\sum\nolimits_{j=0}^{m_n-1} B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\left[\left(Q^{m_n-j-1}\left(\widehat{c}_n^L\right) - \left[V^{m_n-j}\right]^+\left(\widehat{c}_n^L,\widehat{c}_n^L\right)\right)\right] \quad (39)\\ &> \varsigma B\left(m_n-1,n-1,F(\widehat{c}_n^L)\right). \quad (40)\end{aligned}$$

where ς is a strictly positive number that does not depend on n. Notice that $\Psi_{m_n,n}\left(\widehat{c}_n^L\right)$ is the right hand side of (37) evaluated at $\widehat{c}_n^L$.

From the definition of $\widehat{c}_n^L$ we have:

$$\frac{B\left(m_n-2,n-1,F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)}=\frac{m_n}{n-m_n}\frac{1-F(\widehat{c}_n^L)}{F(\widehat{c}_n^L)}=\frac{\frac{n}{L}-m_n}{n-m_n}\to\frac{1}{L}$$

as $n\to\infty$. Similarly, one obtains, for $j=2,...,m_n-1$:

$$\frac{B\left(m_n-1-j,n-1,F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)}\to\left(\frac{1}{L}\right)^j$$

as $n\to\infty$. So, for large values of L, the probability of exactly j volunteers, conditional on having less than or equal to m_n-1 volunteers becomes highly concentrated on $j=m_n-1$.

Define $\overline{B}_j^L$ as the probability of exactly $j\le m_n-1$ volunteers, conditional on having less than or equal to m_n-1 volunteers, when the cutpoint is $\widehat{c}_n^L$:

$$\overline{B}_j^L=\frac{B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1}B\left(k,n-1,F\left(\widehat{c}_n^L\right)\right)}$$

Hence, we have

$$\begin{aligned}
1 &= \sum_{j=0}^{m_n-1}\overline{B}_j^L=\sum_{j=0}^{m_n-1}\frac{B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1}B\left(k,n-1,F\left(\widehat{c}_n^L\right)\right)} && (41)\\
&= \sum_{j=0}^{m_n-1}\frac{B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)}{B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)}\cdot\frac{B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)}{\sum_{k=0}^{m_n-1}B\left(k,n-1,F\left(\widehat{c}_n^L\right)\right)} && (42)\\
&\to \sum_{j=0}^{m_n-1}\left(\frac{1}{L}\right)^j\overline{B}_{m_n-1}^L=\overline{B}_{m_n-1}^L\frac{(1-(\frac{1}{L})^{m_n})}{1-\frac{1}{L}}. && (43)
\end{aligned}$$

This implies that there is a n_L sufficiently large such that for $n>n_L$:

$$\overline{B}_{m_n-1}^L>\frac{1}{1+\epsilon}\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} \qquad (44)$$

for all $\epsilon>0$. That is, $\overline{B}_{m_n-1}^L$ approaches 1 for large L.

Next observe that:

$$\begin{aligned}
\Psi_{m_n,n}\left(\widehat{c}_n^L\right) &= e^{-\gamma\Delta}\sum_{j=0}^{m_n-1}B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\left[Q^{m_n-j-1}\left(\widehat{c}_n^L\right)-\left[V^{m_n-j}\right]^+\left(\widehat{c}_n^L,\widehat{c}_n^L\right)\right]\\
&\ge e^{-\gamma\Delta}\left(1-e^{-\gamma\Delta}\right)vB\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)-e^{-\gamma\Delta}\sum_{j=0}^{m_n-2}B\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)v
\end{aligned}$$

since for all c:

$$\begin{aligned}
e^{-\gamma\Delta}\left[Q^0(c)-\left[V^1\right]^+(c,c)\right] &= v-e^{-\gamma\Delta}\left[V^1\right]^+(c,c)\ge\left(1-e^{-\gamma\Delta}\right)v\\
&\text{and}\\
e^{-\gamma\Delta}\left[Q^{m_n-j-1}(c)-\left[V^{m_n-j}\right]^+(c,c)\right] &\ge -e^{-\gamma\Delta}v\text{ for }j=0,...m_n-2
\end{aligned}$$

Substituting the inequality (44), it follows that for $n>n_L$:

$$\Psi_{m_n,n}\left(\widehat{c}_n^L\right)\ge\sum_{j=0}^{m_n-1}vB\left(j,n-1,F\left(\widehat{c}_n^L\right)\right)\cdot\left[\begin{array}{c}\left(1-e^{-\gamma\Delta}\right)\frac{1}{1+\epsilon}\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}}\\ -e^{-\gamma\Delta}\left(1-\frac{1}{1+\epsilon}\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}}\right)\end{array}\right]$$

for all $\epsilon > 0$. For any discounting parameters $\gamma\Delta$, we can choose value of L large enough so that:

$$\left(1-e^{-\gamma\Delta}\right)\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} > e^{-\gamma\Delta}\left(1-\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}}\right) \Leftrightarrow \frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} > e^{-\gamma\Delta} \tag{45}$$

since the right hand side of the first line in (45) is strictly less than 1 and the left hand side converges to 1 as L increases. It follows that for such values of L we have: for $n > n_L$

$$\begin{aligned}\Psi_{m_n,n}\left(\widehat{c}_n^L\right) &\geq \left(1-e^{-\gamma\Delta}\right) v\frac{1-\frac{1}{L}}{1-\left(\frac{1}{L}\right)^{m_n}} B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right) \\ &= \frac{\varsigma}{1-\left(\frac{1}{L}\right)^{m_n}} B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right) > \varsigma B\left(m_n-1,n-1,F\left(\widehat{c}_n^L\right)\right)\end{aligned}$$

for all $n > n_L$, where $\varsigma = \left(1-e^{-\gamma\Delta}\right) v(1-\frac{1}{L})$ is the desired strictly positive constant.

Step 3. From the definition of $\widehat{c}_n^L = F^{-1}(L\frac{m_n}{n})$ and Step 1, we have that for n sufficiently large:

$$\rho_{L,n} < F\left[vB\left(m_n-1,n-1,\rho_{L,n}\right)\right] \Leftrightarrow \widehat{c}_n^L < vB\left(m_n-1,n-1,F(\widehat{c}_n^L)\right) \tag{46}$$

$$\Leftrightarrow F^{-1}\left(L\frac{m_n}{n}\right) = \widehat{c}_n^L \leq \max_{c\in[0,1]}\left[c\,|c \leq \Psi_{m_n,n}\left(c\right)\right] = \overline{c}_n^{m_n}(0) \tag{47}$$

where $\rho_{L,n} = L\alpha_n$, and therefore: $F(\overline{c}^{m_n}(0)) > L\frac{m_n}{n}$.

Note now that, as proven in Theorem 3, the set of possible continuation values is a non empty, closed valued and upperhemicontinuous correspondence in c, so $\Psi_{m_n,n}(c)$ has these properties as well. Let $\underline{\varphi}_{m_n,n}(c)$ and $\overline{\varphi}_{m_n,n}(c)$ be the the minimal and maximal values that can be assumed by $\Psi_{m_n,n}(c)$ in equilibrium. Since $\Psi_{m_n,n}(1) = 0 < 1$ and we have just proven above there is a $\overline{c}^{m_n}(0)$ such that $\Psi_{m_n,n}(\overline{c}^{m_n}(0)) \geq \overline{c}^{m_n}(0)$, there must be a $c_{1,n}^{m_n}(0) \geq F^{-1}(L\frac{m_n}{n})$ such that $\underline{\varphi}_{m_n,n}\left(c_{1,n}^{m_n}(0)\right) \geq c_{1,n}^{m_n}(0)$ and $\underline{\varphi}_{m_n,n}\left(c_{1,n}^{m_n}(0)\right) \leq c_{1,n}^{m_n}(0)$. Since, as also proven in Theorem 3, the set of continuation value functions is convex valued in c, we must also have that $c_{1,n}^{m_n}(0) \in \Psi_{m_n,n}\left(c_{1,n}^{m_n}(0)\right)$. We conclude that $c_{1,n}^{m_n}(0)$ solves (37) and satisfies $F\left(c_{1,n}^{m_n}(0)\right) \geq L\frac{m_n}{n}$. ■

8.10 Proof of Theorem 5

We proceed in three steps.

Step 1. If $m_n = m$ for all n, then $\lim_{n\to\infty}\frac{F(c_{n,1}^m(0))}{m/n} = \infty$. Assume by contradiction that $\frac{nF(c_{n,1}^m(0))}{m} \to L < \infty$. In this case, it can be proven using standard methods that:

$$B(m-1,n-1,F(c_{n,1}^m(0))) \cong \binom{n-1}{m-1}\frac{\left[\left(\frac{m}{n}\right)^{\frac{m}{n}}(1-\frac{m}{n})^{1-\frac{m}{n}}\right]^n}{\frac{m}{n}} \cong \sqrt{\frac{1}{2\pi m(1-\frac{m}{n})}}$$

where the second step follows from the Strirling approximation formula, and "$\cong$" means that left hand side converges to zero or diverges to infinity at the same rate. Since $\frac{nF(c_{n,1}^m(0))}{m} \to L$ implies that $c_{n,1}^m(0) \to \frac{L}{f(0)}\frac{m}{n}$, so we must have that for sufficiently large n:

$$1 \geq \frac{f(0)B(m-1,n-1,F(c_{n,1}^m(0)))}{c_{n,1}^m(0)} \simeq \sqrt{\frac{1}{2\pi\left(\frac{m}{n}\right)^3(1-\frac{m}{n})n}} = \sqrt{\frac{n^2}{2\pi m^3(1-\frac{m}{n})}} \to \infty,$$

a contradiction. We therefore conclude that in equilibrium if m is constant, then: $\frac{nF(c^{m}_{n,1}(0))}{m} \to \infty$.

Step 2. We now prove that if $m_n \to \infty$ as $n \to \infty$ but $m_n \prec n^{2/3}$, then $\lim_{n\to\infty} \frac{F(c^{m_n}_{n,1}(0))}{\alpha_n} \to L > 1$, where L is either bounded but strictly larger than 1 or infinite (and as, defined in the text, $\alpha_n = m_n/n$). Since $\frac{F(c^{m_n}_{n,1}(0))}{\alpha_n} \geq 1$ by Lemma 6, assume by way of contradiction that $\frac{F(c^{m_n}_{n,1}(0))}{\alpha_n} \to 1$. As in Step 1, a standard approximation gives us:

$$B(\alpha_n n - 1, n-1, F(c^{m_n}_{n,1}(0))) \simeq \sqrt{\frac{1}{2\pi\alpha_n(1-\alpha_n)n}}$$

Similarly as in Step 1, by the definition of $c^{m_n}_{n,1}(0))$ and the fact that $m_n \prec n^{2/3}$, we must have, for large n:

$$1 \geq \frac{f(0)B(m_n - 1, n-1, F(c^{m_n}_{n,1}(0)))}{c^{m_n}_{n,1}(0)} \simeq \sqrt{\frac{1}{2\pi\left(\alpha_n\right)^3(1-\alpha_n)n}} \to \infty,$$

a contradiction. We must therefore have that in equilibrium: $\frac{F\left(c^{m_n}_{n,1}(0)\right))}{\alpha_n} \to L > 1$, with L possibly arbitrarily large.

Step 3. We finally prove that if $m_n = m$, a constant, or if $m_n \to \infty$ as $n \to \infty$ but $m_n \prec n^{2/3}$, then the probability of success in the first period converges to 1. Define for convenience here, $\zeta_n = \frac{F(c^{m_n}_{n,1}(0))}{\alpha_n}$. Note that the probability of failure in the first period is equal to the probability that the number of volunteers in period 1, j, is less than or equal to $\alpha_n n$ agents, which can be bounded above:

$$\begin{aligned}
\Pr(j \leq \alpha_n n) &= \Pr(\frac{j}{n} \leq \alpha_n) = \Pr(\frac{j}{n} \leq F\left(c^{m_n}_{n,1}(0)\right) - (F\left(c^{m_n}_{n,1}(0)\right) - \alpha_n)) \\
&\leq \Pr\left[\left|\frac{j}{n} - F\left(c^{m_n}_{n,1}(0)\right)\right| \geq \alpha_n(\zeta_n - 1)\right] \\
&= \Pr\left[\left|\frac{j}{n} - F\left(c^{m_n}_{n,1}(0)\right)\right| \geq \sigma_{c^{m}_{n,1}(0)}(\frac{j}{n}) \cdot \frac{\sqrt{n\alpha_n}(\zeta_n - 1)}{\sqrt{\zeta_n(1 - F\left(c^{m}_{n,1}(0)\right))}}\right] \leq \left(\frac{\sqrt{\zeta_n(1-F\left(c^{m_n}_{n,1}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n - 1)}\right)^2
\end{aligned} \tag{48}$$

where in the second line we used $F\left(c^{m_n}_{n,1}(0)\right) - \alpha_n = \alpha_n(\zeta_n - 1)$; in the third line we define $\sigma_{c^{m_n}_{n,1}(0)}(\frac{j}{n}) = \frac{\sqrt{F\left(c^{m_n}_{n,1}(0)\right)(1-F\left(c^{m_n}_{n,1}(0)\right))}}{\sqrt{n}}$ and used Chebyshev's inequality. We now have two cases to consider. If $m_n = m$, a constant, then by Step 1 we have $\zeta_n \to \infty$ and can rewrite (48) as:

$$\Pr(j \leq \alpha_n n) \leq \left(\frac{\sqrt{\zeta_n(1-F\left(c^{m}_{n,1}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n - 1)}\right)^2 = \lim_{n\to\infty} \frac{1}{\zeta_n}\frac{1}{m(1-\frac{1}{\zeta_n})^2} = 0$$

If instead $m_n \to \infty$ as $n \to \infty$ but $m_n \prec n^{2/3}$, we have by Step 1 that $\zeta_n \to L > 1$ and:

$$\Pr(j \leq \alpha_n n) \leq \left(\frac{\sqrt{\zeta_n(1-F\left(c^{m_n}_{n,1}(0)\right))}}{\sqrt{n\alpha_n}(\zeta_n - 1)}\right)^2 = \lim_{n\to\infty} \frac{1}{m_n}\frac{L}{(L-1)^2} = 0$$

In both cases, we conclude that the probability of failure converges to zero. ■

8.11 Proof of Theorem 6

We proceed in two parts.

Part 1. We first prove that for any PBE, there is a payoff equivalent (static) Honest and Obedient (HO) direct mechanism that achieves an expected utility for each type that is equal to that type's expected payoff in the PBE.

To this goal, first recall that an HO direct mechanism is an *activity function* $\mu : [0,1]^n \to \Delta(2^I)$, that maps each profile of types, $\mathbf{c}$, to a probability distribution over subgroups of agents who volunteer. Note that for any history h_t with $k_{h_t} = k$ and $l_{h_t} = l$ and public signals $\theta_t = (\theta^1, ..., \theta^t)$, the PBE cutpoint at that history can be written as a function $c(h_t) = c^k(l, \theta_t)$, where in case of multiple equilibria the realization of θ_t may allow the players to coordinate on a continuation equilibrium. For any profile of types $\mathbf{c} = (c_1, ...c_n)$, vector of signals θ_t, and lower bound $l = 0$, we can therefore define a PBE sequence of cutpoints $c_t(\mathbf{c},\theta_t)$ as follows. First, define $c_1(\mathbf{c},\theta_1) = c^m(0, \theta_1)$ and let $k_t(\mathbf{c},\theta_t)$ denote the number of missing volunteers at the end of period t, along the equilibrium path, when the type profile is $\mathbf{c}$ and the public signals up to t are θ_t. For $t = 1$, we have:

$$k_1(\mathbf{c},\theta_1) = \max \{0, m - |\{i|c_i \in [0, c_1(\mathbf{c},\theta_1)]\}|\} .$$

Second, for $t > 1$, if $k_t(\mathbf{c},\theta_t) = 0$, then $c_{t+1}(\mathbf{c},\theta_{t+1}) = c_t(\mathbf{c},\theta_t)$; if instead $k_t(\mathbf{c},\theta_t) \geq 1$, then $c_{t+1}(\mathbf{c},\theta_{t+1}) = c^{k_t(\mathbf{c},\theta_t)}(c_t(\mathbf{c},\theta_t), \theta_{t+1})$.

Next, define a direct HO mechanism as follows. For any profile $\mathbf{c}$ and corresponding PBE thresholds, $(c_t(\mathbf{c},\theta_t))_{t=0}^{\infty}$, define $T_i(\mathbf{c}, \theta)$ for each i as the period at which i would volunteer in the PBE when the profile of types is $\mathbf{c}$, if there is such a period. That is:

$$\begin{aligned} T_i(\mathbf{c}, \theta) &= \min \{t|c_t(\mathbf{c},\theta_t) \geq c_i\} \text{ if } \exists t \text{ such that } c_t(\mathbf{c},\theta_t) \geq c_i \\ &= \infty \text{ otherwise} \end{aligned}$$

where $\theta = (\theta^\tau)_{\tau=1}^{\infty}$; and let $S(\mathbf{c}, \theta)$ denote the period at which the game ends with success if there is ever success at $\mathbf{c}$. That is:

$$\begin{aligned} S(\mathbf{c}, \theta) &= \min \{t|k_t(\mathbf{c},\theta_t) = 0\} \text{ if } \exists t \text{ such that } k_t(\mathbf{c},\theta_t) = 0 \\ &= \infty \text{ otherwise} \end{aligned}$$

Denote by $I_t(\mathbf{c}, \theta) = \{i|T_i(\mathbf{c}, \theta) \leq t\}$ the set of agents who have activated up to and including t. We can now define the activity function μ^{DYN} for a static mechanism as follows, where, for each subset of agents, $g \subseteq I$, $\mu_g^{DYN}(\mathbf{c})$ specifies the probability that only the agents in g are activated when the reported cost profile is $\mathbf{c}$:

$$\mu_g^{DYN}(\mathbf{c}) = \begin{cases} \int_\theta \left[\sum_{\{\tau|I_\tau(\mathbf{c},\theta)=g\}} \left(1 - e^{-\gamma\Delta}\right) e^{-\gamma\Delta(\tau-1)}\right] d\Pi(\theta) & |g| < m \\ \int_\theta \left[1_{\{\theta|I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta)=g\}} \cdot e^{-\gamma\Delta(S(\mathbf{c},\theta)-1)}\right] d\Pi(\theta) & \text{for } |g| \geq m \end{cases} \tag{49}$$

where $\Pi(\theta)$ is the distribution of the public signals; and $1_{\{\theta|I_{S(\mathbf{c},\theta)}(\mathbf{c},\theta)=g\}}$ is the indicator function equal to 1 when θ is such that $I_{S(\mathbf{c},\theta)}(\mathbf{c}, \theta) = g$, and zero otherwise. The activity function for the static mechanism, $\mu_g^{DYN}(\mathbf{c})$, is constructed from the PBE by the following multi-step algorithm. When profile $\mathbf{c}$ is reported, in Step 1 all individuals with a type below $c_1(\mathbf{c}, \theta_1) = c^m(0, \theta_1)$ are asked to volunteer (i.e., the set $I_1(\mathbf{c},\theta)$). If there are at least m such individuals, i.e., $k_1(\mathbf{c},\theta_1) = 0$ and $S(\mathbf{c}) = 1$, then the public good is provided and the algorithm stops without proceeding to Step 2. In this case, $S(\mathbf{c},\theta) = 1$ and $\mu^{DYN}_{I_1(\mathbf{c},\theta)}(\mathbf{c}) = 1$. If $k_1(\mathbf{c},\theta_1) > 0$, i.e., $S(\mathbf{c},\theta) > 1$, then with probability $1 - e^{-\gamma\Delta}$ the algorithm also stops without proceeding to Step 2 (and the public good is not provided). In this case, $S(\mathbf{c},\theta) > 1$ and $\mu^{DYN}_{I_1(\mathbf{c},\theta)}(\mathbf{c}) = 1 - e^{-\gamma\Delta}$, as in (49). With probability $e^{-\gamma\Delta}$, instead, the algorithm proceeds to Step 2. In Step 2, a public signal θ^2 is drawn; a cutpoint $c_2(\mathbf{c},\theta_2)$ is determined; and all individuals with a type in the interval $(c_1(\mathbf{c},\theta_1), c_2(\mathbf{c},\theta_2)]$ where $c_2(\mathbf{c},\theta_2) = c^{k_1(\mathbf{c},\theta_1)}(c_1(\mathbf{c},\theta_1),\theta_2)$ are asked to volunteer and the process continues. In general, at any step t at which the algorithm has not yet stopped, individuals with a type in the interval $(c_{t-1}(\mathbf{c},\theta_{t-1}), c_t(\mathbf{c},\theta_t)]$ are asked to volunteer. If there are at least $k_{t-1}(\mathbf{c},\theta_t)$ such individuals, i.e., $k_t(\mathbf{c},\theta_t) = 0$ and $S(\mathbf{c},\theta) = t$, then the public good is provided in Step t and the algorithm stops selecting $I_t(\mathbf{c},\theta)$ without proceeding to Step $t+1$. If $k_t(\mathbf{c},\theta_t) > 0$, i.e., $S(\mathbf{c},\theta_t) > t$, then with probability $1-e^{-\gamma\Delta}$ the algorithm also stops without proceeding to step $t+1$ (and the public good is not provided), and with probability $e^{-\gamma\Delta}$ the algorithm proceeds to step $t+1$. In all cases, the probabilities are given at each step by (49). Thus, the static mechanism mimics the discounting in the dynamic game by randomly stopping the algorithm with probability $1-e^{-\gamma\Delta}$ after any step at which the threshold m has not yet been achieved.

From the above construction of μ^{DYN}_g, we can represent the probability of success and that a player i is asked to volunteer by:

$$\begin{aligned} P(\mathbf{c}) &= \int_\theta e^{-\gamma\Delta(S(\mathbf{c},\theta)-1)} d\Pi(\theta) \\ A_i(\mathbf{c}) &= \int_{\{\theta | S(\mathbf{c},\theta) \geq T_i(\mathbf{c},\theta)\}} e^{-\gamma\Delta(T_i(\mathbf{c},\theta)-1)} d\Pi(\theta) \end{aligned}$$

where $P(\mathbf{c})$ is the probability of obtaining the public good at profile $\mathbf{c}$, and $A_i(\mathbf{c})$ is the probability that i is asked to volunteer at $\mathbf{c}$. The expected utility for an individual with type c at profile $\mathbf{c} = (c, \mathbf{c}_{-i})$ is

$$U_i(\mathbf{c}) = vP(\mathbf{c}) - c_i A_i(\mathbf{c})$$

This is exactly equal to the expected utility for an individual with type c_i at profile $\mathbf{c}$ in the corresponding PBE of the dynamic game. We only need to prove that this direct mechanism is Honest and Obedient (Myerson, 1982). We need to show that every type c is weakly better off reporting c and obeying all recommendations, than they would be reporting c and disobeying some recommendations or reporting $c' \neq c$ and then following some optimal strategy in terms of obedience/non-obedience of the subsequent recommendation.

Suppose there is a player i of type c who is strictly better off reporting to be a type $c' > c$. The analysis of the case in which i reports be a type $c' < c$ is analogous and omitted. There are two cases, corresponding to the two information sets in which i can find himself/herself: when the recommendation is to volunteer; and when it is to not volunteer. We proceed in 2 steps.

Step 1. Consider first the case in which the recommendation is to volunteer.

Step 1.1. We first show that if by reporting c' the recommendation is to volunteer, then the same recommendation must be received by reporting c. Since c' has received a recommendation to volunteer, it must be that $\mathbf{c}_{-i}$ is such that $c_t(c', \mathbf{c}_{-i}, \theta_t) \geq c'$ for some some $t \leq S(\mathbf{c},\theta)$ and a sequence of cutoffs $c_t(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. Let t' be the smallest period in which $c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$, then $c \in I_{t'}(c', \mathbf{c}_{-i}, \theta)$. Note, moreover, that by definition $c_{t''}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and $c_{t'}(\widetilde{c}, \mathbf{c}_{-i}, \theta_{t'})$ is the same if $\widetilde{c} = c'$ or $\widetilde{c} = c$, so $c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t'}(c, \mathbf{c}_{-i}, \theta_{t'})$ and $I_{t'}(c', \mathbf{c}_{-i}, \theta) = I_{t'}(c, \mathbf{c}_{-i}, \theta)$. Since $t' \leq t$, we have $I_{t'}(c, \mathbf{c}_{-i}, \theta) = I_{t'}(c', \mathbf{c}_{-i}, \theta) \subseteq I_t(c', \mathbf{c}_{-i}, \theta)$: we conclude that a recommendation to volunteer to a type c', implies the same recommendation to a player who reports to be a type c as well. When the recommendation is to volunteer and the player obeys, then reporting $c' > c$ cannot be strictly superior than reporting c.

Step 1.2. Assume now the recommendation is to volunteer and the player strictly prefers to disobey by not volunteering. In this case, again, $\mathbf{c}_{-i}$ must be such that $c_t(c', \mathbf{c}_{-i}, \theta_t) \geq c'$ for some $t \leq S(\mathbf{c})$ and a sequence of cutoffs $c_t(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. As in Step 1.1, let t' be the minimal period in which $c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Then $c_{t''}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and so $c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t'}(c, \mathbf{c}_{-i}, \theta_{t'})$. If $t' > t$, then $c_t(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_t(c, \mathbf{c}_{-i}, \theta_t) < c$ for $t \leq S(\mathbf{c}, \theta)$, so if the agent reported truthfully, s/he would have received the recommendation to not volunteer. It follows that in this event reporting c' and disobeying induces the same action as reporting c and obeying: it cannot generate a strictly superior deviation in this event. If instead, $t' \leq t$, then $c_t(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t'}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Player i does not know $\mathbf{c}_{-i}$ and t, but s/he knows that conditioning on being asked to volunteer, $\mathbf{c}_{-i}$ is such that there is a $t \leq S(\mathbf{c}, \theta)$ in which $c_t(c, \mathbf{c}_{-i}, \theta_t) \geq c$. This implies the following. First that player i conditions on an event in which the set $I_{t-1}(c, \mathbf{c}_{-i}, \theta_{t-1})$ of players volunteers for sure (indeed i conditions on a family of events with this property). Second, i conditions on an event in which, for any $j \geq 0$, the cutoffs at $t+j$ are identical to the cutoffs in the PBE of the dynamic game (by construction) that follows the cutoff $c_t(c, \mathbf{c}_{-i}, \theta_t)$. It follows that i has the same expected values as in the PBE, and s/he weakly prefers to volunteer: s/he therefore cannot strictly prefer to disobey the mechanism and not volunteer. We conclude that if the player disobeys when asked to volunteer after reporting to be a type c', the deviation cannot be strictly superior that reporting honestly and then obeying the recommended action.

Step 2. Consider now the case in which the recommendation is to abstain.

Step 2.1. Consider first the case in which i reports c' and obeys to a recommendation to abstain. In this case, again, $\mathbf{c}_{-i}$ must be such that $c_t(c', \mathbf{c}_{-i}, \theta_t) \leq c'$ for some $t \leq S(\mathbf{c}, \theta)$ and a sequence of cutoffs $c_t(c', \mathbf{c}_{-i}, \theta_t)$ corresponding to a sequence of public signals θ_t, followed in a PBE with positive probability. Let t' be the minimal period in which $c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Then $c_{t''}(c', \mathbf{c}_{-i}, \theta_{t''}) < c$ for all $t'' < t'$, and so by the same argument as in Step 1, $c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) = c_{t'}(c, \mathbf{c}_{-i}, \theta_{t'})$. If $t' > t$, then $c_t(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_t(c, \mathbf{c}_{-i}, \theta_t) < c$, so in this event reporting c' induces the same action as reporting c: it cannot generate a strictly superior deviation in this event.

If instead, $t' \leq t$, then $c_t(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t'}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. Player i does not know $\mathbf{c}_{-i}$ and t, but s/he knows that conditioning on being asked to volunteer, $\mathbf{c}_{-i}$ is such that there is a $t \leq S(\mathbf{c}, \theta)$ in which $c_t(c, \mathbf{c}_{-i}, \theta_t) \geq c$. As in Step 1.2, this implies that i has the same expected values as in the PBE, and weakly prefers to volunteer. So reporting c' and obeying a recommendation to abstain cannot yield a higher expected utility than reporting truthfully and obeying the recommendation of the mechanism.

Step 2.2. Finally, consider the case in which i reports c' and disobeys to a recommendation to abstain. Again, let t' be the minimal period in which $c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$. If $t' > t$, then $c_t(c', \mathbf{c}_{-i}, \theta_t) < c'$ and $c_{t'}(c, \mathbf{c}_{-i}, \theta_{t'}) < c$ for all $t' \leq t$. So a type c would find it optimal to abstain. If instead, $t' \leq t$, then $c_t(c, \mathbf{c}_{-i}, \theta_t) \geq c_{t'}(c, \mathbf{c}_{-i}, \theta_{t'}) = c_{t'}(c', \mathbf{c}_{-i}, \theta_{t'}) \geq c$, and a type c would receive the same expected payoff from reporting truthfully and obeying than from reporting c' and disobeying a recommendation to abstain.

Since there is no scenario in which the player finds it strictly optimal to report to be a type $c' > c$, we conclude that the player is never strictly better off by reporting to be $c' > c$, no matter what obedience policy s/he follows afterwards.

Part 2. We now prove that Part 1 implies the result. From Part 1 we know that the expected utility obtained in the PBE is equal to the expected utility obtained in a specific direct, static mechanism that is honest and obedient. Battaglini and Palfrey (2024) have proven that the expected utility of a player in the best direct, static mechanism that is honest and obedient converges to 0 as $n \to \infty$ when $m_n \succ n^{2/3}$ or, equivalently, when $\alpha_n / \sqrt[3]{1/n} \to \infty$ (See Theorem 4). It follows that he same must be true in the PBE. ■

8.12 Proof of Part 2 of Theorem 4'

To simplify notation, we suppress the dependence of the lower bound of the posterior beliefs, l, on h_t^k. For any lower bound, l, define $\underline{c}^k(l)$ as the minimal x such that:

$$\begin{aligned} & v - x - e^{-\gamma\Delta} \cdot \sum_{j=0}^{k-2} \left(\frac{v}{e^{-\gamma\Delta}} - Q^{k-j-1}(x) \right) B\left(j, n-1-m+k, \widetilde{F}(x;l)\right) \\ & \geq v - e^{-\gamma\Delta} \cdot \sum_{j=0}^{k-1} \left(\frac{v}{e^{-\gamma\Delta}} - V^{k-j}(x,x) \right) B\left(j, n-1-m+k, \widetilde{F}(x;l)\right). \end{aligned} \tag{50}$$

The left hand side is the utility of a cutpoint type x who volunteers when the cutpoint used by the other agents is x. The right hand side is the utility of a type x who does not volunteer, when the others are using cutpoint x. Note that the left hand side may be strictly lower than the right hand side for any $x \in [l, v]$: in this case all types c strictly prefer not to volunteer and $c^k(l) = l$, in which case the cutpoint is not defined by an equality as in (50). It follows that there is no type $x < v$ that is willing to volunteer; when $\underline{c}^k(l) > l$, then any type $x \leq \underline{c}^k(l)$ is willing to volunteer. When $\underline{c}^k(l) = l$, then type l is willing to volunteer only (50) holds with equality.

We can write (50) as:

$$\underline{c}^k(l) = \min_{c \geq [l,1]} \left\{ c | c \leq e^{-\gamma\Delta} \sum_{j=0}^{k-1} \left[\left(Q^{k-j-1}(c) - V^{k-j}(c,c) \right) B\left(j, n-1-m+k, \widetilde{F}(c;l)\right) \right] \right\} \tag{51}$$

where $Q^0(c) = v/e^{-\gamma\Delta}$. We now prove that there is a $v^*(n,m,\gamma,\Delta)$ such that for $v > v^*(n,m,\gamma,\Delta)$, $c^k(l) > l$ for any $k \leq m$ and $l < \min\{v, 1\}$. We proceed in four steps.

Step 1. We have already proven in Lemma 2 that for any $l < v$, we have $\underline{c}^1(l) > l$. Moreover it is easy to verify that there must be a $v^*(n,1,\gamma,\Delta)$ such that for $v \geq v^*(n,1,\gamma,\Delta)$ we have $e^{-\gamma\Delta}Q^1(l) - l > 0$ for any $l \in [0, \min\{v,1\}]$: since as it can be verified using (4), $c_t^1(l)$ is strictly increasing in v for all t; and $Q^1(l)$ is increasing in both v and $c_t^1(l)$ for all t.

Step 2. For the induction hypothesis, assume that for any $l \in [0, \min\{v,1\}]$ and for all $j = 1, ..., m-1$ there is a $v^*(n,j,\gamma,\Delta) > 0$ such that for $v \geq v^*(n,j,\gamma,\Delta)$, we have: $c^j(l) > l$ and

$e^{-\gamma\Delta}Q^{j}(l) - l > 0$. We prove that there is a $v^*(n, m, \gamma, \Delta)$ such that for $v \geq v^*(n, m, \gamma, \Delta)$, we have: $c^j(l) > l$ for all $l < v$ and $e^{-\gamma\Delta}Q^{j}(l) - l > 0$ for any $j \leq m$ and $l \in [0, 1]$. There are two sub-cases to consider.

Step 2.1. Suppose by contradiction that for any v even arbitrarily large, $c^k(l) = l$ and we have that $\left[V^k\right]^{-}(c^k(l), l) > \left[V^k\right]^{+}(l, l)$ for some $l \leq \min\{1, v\}$. Hence, at l we therefore have a strict corner solution when there are k missing volunteers. In this case, the value function for a type l must be $V^k(l, l) = 0$, since the project will never be realized: all players expect that no other player of type $c \geq l$ is willing to contribute. Suppose that $v \geq v^*(n, k-1, \gamma, \Delta)$, as defined by the induction step. If a player of type l volunteers, s/he obtains: $\left[V^k\right]^{+}(l, l) \geq e^{-\gamma\Delta} \cdot Q^{k-1}(l) - l > 0$, where the last inequality follows from the induction step: we thus have a contradiction.

Step 2.2. From the previous step, we conclude that, if Part 1 the theorem is not true, then for $v > v^*(n, k-1, \gamma, \Delta)$, if $c^k(l) = l$ then $V^k(c^k(l), l) = \left[V^k\right]^{+}(l, l)$. From $c^k(l) = l$, we have:

$$\begin{aligned} B\left(j, n-1-m+k, \widetilde{F}(c; l)\right) &= 0 \text{ for } j > 0 \\ B\left(0, n-1-m+k, \widetilde{F}(c; l)\right) &= 1 \end{aligned} \tag{52}$$

since $V^k(c^k(l), l) = \left[V^k\right]^{+}(l, l)$. Hence $\underline{c}^k(l)$ satisfies (51) at equality, which implies that (51) can be written as: $l = \underline{c}^k(l) = e^{-\gamma\Delta} \cdot \left(Q^{k-1}(l) - \left[V^k\right]^{+}(l, l)\right)$. Note that when $\underline{c}^k(l) = l$, there are no other volunteers, so: $\left[V^k\right]^{+}(l, l) = e^{-\gamma\Delta} \cdot Q^{k-1}(l) - l$. But then we have:

$$\begin{aligned} Q^{k-1}(l) - \left[V^k\right]^{+}(l, l) &= Q^{k-1}(l) - \left(e^{-\gamma\Delta} \cdot Q^{k-1}(l) - l\right) \\ &= l + (1 - e^{-\gamma\Delta})Q^{k-1}(l) \end{aligned}$$

This implies that we have $e^{-\gamma\Delta} \cdot \left(Q^{k-1}(l) - \left[V^k\right]^{+}(l, l)\right) > l$ if $e^{-\gamma\Delta}l + e^{-\gamma\Delta}(1 - e^{-\gamma\Delta})Q^{k-1}(l) > l$: or equivalently $e^{-\gamma\Delta}Q^{k-1}(l) > l$, an inequality that is always true if $v > v^*(n, k-1)$. But then we have $l = \underline{c}^k(l) = e^{-\gamma\Delta} \cdot \left(l + (1 - e^{-\gamma\Delta})Q^{k-1}(l)\right) > l$, a contradiction. We conclude that for for any $k \leq m$ and $l \leq \min\{1, v\}$, $c^k(l) > l$ if $v > v^*(n, k, \gamma, \Delta)$.

Step 3. Finally, we conclude the inductive argument by proving that there is a $v^*(n, k, \gamma, \Delta) \geq v^*(n, k-1, \gamma, \Delta)$ such that for $v > v^*(n, k, \gamma, \Delta)$, then $e^{-\gamma\Delta}Q^k(l) > l$ for any $l \in [l_0, \min\{1, v\}]$. It is sufficient to prove $e^{-\gamma\Delta}Q^k(l) > 1$, for v sufficiently high. Assume not. Then it must be that $c^k(l)$ converges to l as v increases, since if it converges to a constant $\widetilde{c} > l$. We must therefore have that $B\left(j, n-1-m+k, \frac{F(\widetilde{c}) - F(l)}{1 - F(l)}\right) > 0$ for all $j \geq k$ and $Q^k(l) \to \infty$ as $v \to \infty$, since it is strictly increasing in v (and diverging at infinity as v increases, given the lower bounds on the probabilities of $j \geq k$ volunteers): a contradiction. But if $c^k(l) \to l$, then we have: $\left[V^k\right]^{+}(l, l) \to e^{-\gamma\Delta}Q^{k-1}(l) - l$. Since, by the previous step, the equilibrium is interior in stage k for $v > v^*(n, k-1, \gamma, \Delta)$, we have: $c^k(l) = e^{-\gamma\Delta} \cdot \left(Q^{k-1}(l) - \left[V^k\right]^{+}(l, l)\right)$. Note moreover that for $v > v^*(n, k-1, \gamma, \Delta)$, we have $e^{-\gamma\Delta}Q^{k-1}(l) > l$. It follows that as v increases, we have:

$$c^k(l) \to e^{-\gamma\Delta} \cdot \left(l + (1 - e^{-\gamma\Delta})Q^{k-1}(l)\right) > e^{-\gamma\Delta} \cdot \left(l + (1 - e^{-\gamma\Delta})\frac{l}{e^{-\gamma\Delta}}\right) = l$$

where the last inequality follows from $v > v^*(n, k-1, \gamma, \Delta)$. We thus have a contradiction. We conclude that there is a $v^*(n, k)$ such that for $v > v^*(n, k, \gamma, \Delta)$, $e^{-\gamma\Delta}Q^k(l) > l$. ■

8.13 Proof of Proposition 2

We show here that if $v > v^*(n, m, \gamma, \Delta)$, then the group achieves the objective if there are at least m players with type lower than v. Note that for $v > v^*(n, m, \gamma, \Delta)$, in the history h_t^k with k volunteers needed and a lower bound of types at $l_{h_t^k}$, then in period $t+1$, there will either be $j < k$ volunteers needed, with a lower bound of $c^k(l_{h_t^k})$; or there will still be k volunteers needed with a higher lower bound of types $c^k(l) > l$.

Now suppose, by way of contradiction, that there is some k for which $c_t^k \to c_\infty^k < v$ for some initial $l_{h_t^k}$, following a sequence of many periods where there are no additional volunteers beyond $m - k$. Note that as $c_t^k \to c_\infty^k$, we have:

$$\widetilde{F}(c_t^k; c_{t-1}^k) \to \frac{\lim_{t\to\infty}\left(F(c_t^k) - F(c_{t-1}^k)\right)}{1 - F(c_\infty^k)} = 0$$

Since (15) must hold, we therefore have:

$$c_\infty^k = \min_{c \in \left[c_\infty^k, 1\right]} \left\{ c \geq e^{-\gamma\Delta} \cdot \left(Q^{k-1}(c) - \left[V^k\right]^+ (c, c) \right) \right\},$$

But then the same argument as Step 2.2. above proves that for $v > v^*(n, m)$ we must have:

$$\min_{c \in \left[c_\infty^k, 1\right]} \left\{ c \geq e^{-\gamma\Delta} \cdot \begin{bmatrix} Q^{k-1}(c) \\ -\left[V^k\right]^+ (c, c) \end{bmatrix} \right\} > e^{-\gamma\Delta} \cdot \left(c_\infty^k + (1 - e^{-\gamma\Delta}) \frac{c_\infty^k}{e^{-\gamma\Delta}} \right) > c_\infty^k,$$

a contradiction. We conclude that for all k, $c_t^k \to c_\infty^k = v$. ■

8.14 Proof of Proposition 3

Call E^{2+} the event comprising histories h_t in which at least two volunteers are missing and they both have a cost $c_i \in (v/2, 1)$. Clearly this event has positive probability for any $v \in (1, 2)$. We now prove that for any $v \in (1, 2)$, there is an equilibrium in which contributions are zero in E^{2+}, no matter what the level of γ, and Δ are, thus even in the limit as $\gamma, \Delta \to 0$. Consider an history h_t^2 with the properties as above. Assume the lower-bound on types is $l > v/2$. An active player i who expects no other active player to contribute obtains from contributing at most a payoff

$$\begin{aligned} e^{-\gamma\Delta} Q^1(c^2(l)) - c_i &\leq e^{-\gamma\Delta}\left[v - c^2(l)\right] - c_i \leq e^{-\gamma\Delta}\left[v - l\right] - l \\ &\leq \left[e^{-\gamma\Delta} - \frac{1}{2}(1 + e^{-\gamma\Delta})\right] v < 0 \end{aligned} \tag{53}$$

where the first inequality follows from the fact that $Q^1(c^2(l))$ must be smaller than the utility of the lowest remaining type, i.e. $c^2(l)$, so it must be $Q^1(c^2(l)) \leq v - c^2(l) \leq v - l$; the second inequality follows from the fact that $c^2(l) \geq l$ and $c_i \geq l$. We conclude that no active player finds it optimal to contribute if s/he does not expect some other player to contribute with positive probability. ■

8.15 Proof of Proposition 4

We prove here that for any $n > m_n$ and for any $\varepsilon > 0$, $\gamma \in (0, 1)$ there exists a $\Delta_{n,\varepsilon,\gamma} > 0$ such that for $\Delta > \Delta_{n,\varepsilon,\gamma}$ the project is realized in an equilibrium with probability less than ε. This implies that for any $\widehat{v}$, there is a $\Delta_{\widehat{v}}$ such that for $\Delta > \Delta_{\widehat{v}}$ we have $v^*(n, m.\gamma, \Delta) > \widehat{v}$. For any F and for any $\varepsilon > 0$, there is a $c_\varepsilon > 0$ such that with probability $1 - \varepsilon$, at least $n - m_n + 2$ players have cost strictly larger than c_ε (so that there are no more than $m_n - 2$ players with cost lower than c_ε). Consider a subgame with $k \leq m_n$ and $l \geq c_\varepsilon$, where k is the number of missing volunteers, and l is the lower bound on types. For these subgames, consider the path of future play along which no uncommitted member volunteers. To see that there is a $\Delta_{n,\varepsilon,\gamma}$ such that for $\Delta > \Delta_{n,\varepsilon,\gamma}$ this is an equilibrium, note that with these strategies the expected utility of a player who does not volunteer is zero; the expected benefit of volunteering player is not higher than $D_{n,\varepsilon,\gamma} \equiv -c_\varepsilon + e^{-\gamma\Delta}(v - c_\varepsilon)$: success can occur no sooner than a period after the deviator volunteers, and the expected payoff the period after a unilateral deviation cannot be larger than the utility of the lowest type, i.e. $v - c_\varepsilon$. For $\Delta > \frac{1}{\gamma} \log\left(\frac{v-c_\varepsilon}{c_\varepsilon}\right) = \Delta_{n,\varepsilon,\gamma}$, we have that $D_{n,\varepsilon,\gamma} < 0$, so the equilibrium strategies are optimal. Given these equilibrium strategies, assign to any other k', l' with $k' \leq m_n$ and $l' < c_\varepsilon$, some corresponding equilibrium strategy for the continuation game. In the equilibrium of the overall game it must be that if $\Delta > \Delta_{n,\varepsilon,\gamma}$ then with probability $1 - \varepsilon$ there are not enough members with cost $c < c_\varepsilon$ to complete the project, which then must fail: indeed, if $\Delta > \Delta_{n,\varepsilon,\gamma}$ then with probability at least $1 - \varepsilon$ either we reach a subgame k', l' with $k' \leq m_n$ and $l' < c_\varepsilon$ in which no player contributes; or we reach a state k, l with $k \leq m_n$ and $l \geq c_\varepsilon$, in which case again no player finds it optimal to contribute by construction. ■

8.16 Proof of Proposition 5

We proceed in two steps.

Step 1. Consider period $T > 1$ and assume that the number of missing volunteers is larger or equal than $k_n = \frac{m_n}{T} > 0$; and that the lower bound of types is $l \geq 0$. We now prove that there is $n^{(T)}$ such that for $n > n^{(T)}$, the probability of contributing is zero for all players. It follows that at any history in which $k_n \geq \frac{m_n}{T}$ at stage T, then the continuation values are $Q_T^{k_n}(l) = V_T^{k_n}(c, l) = 0$ for any $c \geq l$ and $l \geq 0$.

At period T, let the equilibrium cutpoint be $c_n^{T,T}$ (we omit here for simplicity the dependence on h_t and l), which must satisfy:

$$c_n^{T,T} = vB\left(\beta_n z_n - 1, z_n - 1, \widetilde{F}(c_n^{T,T}; l)\right) = \Psi\left(c_n^{T,T}\right) \tag{54}$$

where we define the function $\Psi\left(c_n^{T,T}\right)$, and $z_n = (1 - \alpha)n + k_n$ and $\beta_n = \frac{k_n}{z_n} \geq \frac{\alpha}{T} > 0$. We now prove that for n large enough, it must be $c_n^{T,T} = l$. If $l > 0$ and $c_n^{T,T} > l$, then the right hand side of (54) converges to zero, but the left hand side converges to a strictly positive value, a contradiction. Assume therefore that $l = 0$ and $c_n^{T,T} > 0$. Define $\widehat{\beta}_n$ to be the value such that $\widetilde{F}(\widehat{\beta}_n; l) = \frac{\beta_n}{1-1/z_n}$. This is the value that maximizes the right hand side of (54), i.e. $\Psi(\cdot)$. It is straightforward to verify that it must be that $\widehat{\beta}_n > 0$ for any n. Moreover, since $\beta_n \geq \frac{\alpha}{T}$, we have that for n large enough:

$$vB\left(\beta_n z_n - 1, z_n - 1, \frac{\beta_n}{1 - 1/z_n}\right) = vB\left(\beta_n z_n - 1, z_n - 1, \widetilde{F}(\widehat{\beta}_n; l)\right) < \widehat{\beta}_n \simeq \beta_n / f(0), \tag{55}$$

given that the first and second terms converge to zero, but $\widehat{\beta}_n$ converges to $\beta_n/f(0)$, which is strictly positive for all n. From the inequality in 55, we have that $\Psi'\left(c_n^{T,T}\right) < 1$, at any fixed point $c_n^{T,T}$ of

Ψ. This follows from the fact that the right hand side of (54) has a maximum below the 45^o line, so if there is a strictly positive fixed point, there must be a fixed point at which $\Psi(c)$ intersects the 45^o line from above. But, as we now show, this is impossible. To see this note that:

$$B'\left(\beta_n z_n - 1, z_n - 1, F\left(c_n^{T,T}\right)\right) = B\left(\beta_n z_n - 1, z_n - 1, F\left(c_n^{T,T}\right)\right)\left[\begin{array}{c}\frac{\beta_n z_n - 1}{F\left(c_n^{T,T}\right)} \\ -\frac{z_n - \beta_n z_n}{1-F\left(c_n^{T,T}\right)}\end{array}\right] f\left(c_n^{T,T}\right) \tag{56}$$

$$\to \frac{f(0)\, c_n^{T,T}}{v}\left[\frac{\beta_n z_n - 1}{f(0)\, c_n^{T}} - z_n - \beta_n z_n\right] = \frac{1}{v}\left[1 - f(0)\, c_n^{T,T} - f(0)\frac{c_n^{T,T}}{\beta_n} - \frac{1}{\beta_n z_n}\right]\cdot \beta_n z_n \to \infty$$

So we have a contradiction, since the right hand side of (55) converges to a bounded value. We conclude that if $k_n \geq \frac{m_n}{T}$, then $c_n^{T,T} = l$ is the unique fixed point of (54) for any $l \geq 0$.

Step 2. Assume as an induction step that for some $t < T$ and all $\tau \geq t+1$ we have: there is a $n^{(\tau)}$ such that for $n > n^{(\tau)}$ we have $V_\tau^{k_n^\tau}(c, l_n) = Q_\tau^{k_n^\tau}(l_n) = 0$ for all $c \geq 0$ when $k_n^\tau \geq \overline{k}_n^\tau = (T-\tau+1)\frac{m_n}{T}$. This property is true for $t+1 = T$ by Step 1. We prove the result if we prove that:

$$c_n^{t,T} = e^{-\gamma\Delta}\sum\nolimits_{j=0}^{k_n^t - 1}\left[\left(Q^{k_n^t - j - 1}\left(c_n^{t,T}\right) - V^{k_n^t - j}(c_n^{t,T}, c_n^{t,T})\right) B\left(j, n-1-m_n+k_n^t, \widetilde{F}(c_n^{t,T}; l_n)\right)\right]. \tag{57}$$

has a no strictly positive fixed point $c_n^{t,T}$ when $k_n^\tau \geq \overline{k}_n^\tau = (T-\tau+1)\frac{m_n}{T}$. To this goal define, similarly as in Step 1, $z_n^t = n - m_n + k_n^t$, $\beta_n^t = k_n^t / z_n^t$ and $\widetilde{F}_n^t = \widetilde{F}(c_n^{t,T}; l_n)$.

By the induction step we have

$$c_n^{t,T} = e^{-\gamma\Delta}\sum\nolimits_{j=\frac{m_n}{T}+1}^{k_n^t - 1}\left[\left(Q^{k_n^t - j - 1}\left(c_n^{t,T}\right) - V^{k_n^t - j}(c_n^{t,T}, c_n^{t,T})\right) B\left(j, z_n^t - 1, \widetilde{F}_n^t\right)\right] \tag{58}$$

The right hand side of (58) can be bounded above by:

$$\begin{aligned} & e^{-\gamma\Delta} v \cdot \sum\nolimits_{j=\frac{m_n}{T}+1}^{k_n^t - 1}\left[B\left(j, z_n^t - 1, \widetilde{F}_n^t\right)\right] \\ \leq\ & \exp\left(-n\left(\frac{\alpha}{T}\log\frac{\alpha/T}{\widetilde{F}_n^t} + (1-\frac{\alpha}{T})\log\frac{1-\alpha/T}{1-\widetilde{F}_n^t}\right)\right) = D_n(\widetilde{F}_n^t) \end{aligned}$$

where for the inequality we used the Chernoff bound of the upper tail of the Binomial distribution (see, for example, Ash [1990, 4.7.2)]. Without loss of generality we can assume that $\widetilde{F}_n^t < \frac{\alpha}{T}$ for n sufficiently large. Indeed, if this were not the case then we would have some $c^{\alpha/T} > 0$ such that $c_n^{t,T} > c^{\alpha/T}$, but this is impossible in equilibrium since the expected benefit of contributing for a single player converges to zero as $n \to \infty$. Note that for any $\widetilde{F}_n^t > \underline{F}$ for some $\underline{F} > 0$, we have $D_n(\widetilde{F}_n^t) < \underline{F}$, since $D_n(\widetilde{F}_n^t) \to 0$. Moreover a Taylor approximation tells us that for $F < \underline{F}$ with $\underline{F}$ sufficiently small we have: $D_n(\widetilde{F}_n^t) = D_n(0) + D_n'(0)\widetilde{F}_n^t + o(\widetilde{F}_n^t)$, where $o(\widetilde{F}_n^t)/\widetilde{F}_n^t \to 0$ as $\widetilde{F}_n^t \to 0$. But then, if we have a positive fixed $c_n^{t,T}$ point, we have:

$$c_n^{t,T} \leq D_n(0) + D_n'(0) f(0) c_n^{t,T} + o(c_n^{t,T}) = c_n^{t,T}\left[D_n'(0) f(0) + \frac{o(c_n^{t,T})}{c_n^{t,T}}\right] < c_n^{t,T}$$

since $\left[D_n'(0) f(0) + \frac{o(c_n^{t,T})}{c_n^{t,T}}\right]$ can be chosen to be arbitrarily small, a contradiction. We can iterate the argument up to the first period. We must therefore have $m_n \geq \overline{k}_n^1 = (T)\frac{m_n}{T} = m_n$, which implies that $c_n^1 = 0$ for $n > n^{(1)}$. ■

References

[1] Admati, A. and M. Perry (1991), "Joint Projects Without Commitment" *Review of Economic Studies*, 58(2): 259-276.

[2] Alesina A and A. Drazen (1991), "Why are Stabilizations Delayed?" *The American Economic Review*, 81(5): 1170-1188.

[3] Archetti, M. (2009). "The volunteer's dilemma and the optimal size of a social group." *Journal of Theoretical Biology*, 261(3), 475–480. https://doi.org/10.1016/j.jtbi.2009.08.018

[4] Archetti, M., and I. Scheuring (2012). "Game theory of public goods in one-shot social dilemmas without assortment." *Journal of Theoretical Biology*, 299, 9–20.

[5] d'Aspremont, Claude and L.-A. Gérard-Varet (1979), "Incentives and Incomplete Information", *Journal of Public Economics*, 11, 25-45.

[6] d'Aspremont C., J. Crémer, and L.-A. Gérard-Varet (1990), "Incentives and the existence of Pareto-optimal revelation mechanisms", *Journal of Economic Theory*, 51(2): 233-254.

[7] Battaglini M and S. Coate (2007), "Inefficiency in Legislative Policy-Making: A Dynamic Analysis," *American Economic Review*, 97(1): 118-149.

[8] Battaglini M., S. Nunnari and T.R. Palfrey (2014), "Dynamic Free Riding with Irreversible Investments," *American Economic Review*, 104(9): 2858-71.

[9] Battaglini M., S. Nunnari and T.R. Palfrey (2016), "The Dynamic Free Rider Problem: A Laboratory Study," *American Economic Journal: Microeconomics*, 8(4):268-308.

[10] Battaglini M. and T.R. Palfrey (2024), "Organizing for Collective Action: Olson Revisited," *Journal of Political Economy*, Forthcoming.

[11] Bergstrom T. (2017), "Efficient Ethical Rules for Volunteer's Dilemmas", University of California Santa Barbara, mimeo.

[12] Bilodeau, M. and A. Slivinsky (1996), "Toilet cleaning and department chairing: Volunteering a public service," *Journal of Public Economics*, 59 (1996) 299-308.

[13] Bliss, C. and B. Nalebuff (1984), "Dragon Slaying and Ballroom Dancing: The Private Supply of a Public Good ", *Journal of Public Economics*, 25: 1-12.

[14] Bulow, J. and P. Klemperer (1999), "The Generalized War of Attrition" *American Economic Review*, 89(1): 175-189.

[15] Chen, X, T. Gross, U. Dieckmann (2013), "Shared rewarding overcomes defection traps in generalized volunteer's dilemmas", *Journal of Theoretical Biology*, 335: 13–21.

[16] Choi, S., D. Gale, and S. Kariv (2008), "Sequential equilibrium in monotone games: A theory-based analysis of experimental data" *Journal of Economic Theory*, 143(1):302–330.

[17] Choi, S., D. Gale, S. Kariv, and T. R. Palfrey (2011), “Network architecture, salience and coordination” *Games and Economic Behavior*, 73:76–90.

[18] Crémer, J. and R.P. McLean (1988), “Full Extraction of the Surplus in Bayesian and Dominant Strategy Auctions,” *Econometrica*, 56, 1247–1257.

[19] Darley, J. M. and B. Latané (1968). “Bystander intervention in emergencies: Diffusion of responsibility”. *Journal of Personality and Social Psychology.* 8 (4): 377–383.

[20] Diekmann, A. (1985), “Volunteer’s dilemma”, *Journal of Conflict Resolution*, 29:605-610.

[21] Esteban J. and D. Ray (2001), “Collective action and the group paradox”, *American Political Science Review*, 95(3):663–672.

[22] Fershtman C. and S. Nitzan (1991), “Dynamic Voluntary Provision of Public Goods”, *European Economic Review*, 35: 1057-1067.

[23] Fredriksson G and N. Gaston (2000), “Ratification of the 1992 Climate Change Convention: What Determines Legislative Delay?” *Public Choice*, 104(3/4): 345-368.

[24] Fudenberg D. and D. Levine (1983), “Subgame-Perfect Equilibria of Finite- and Infinite-Horizon Games.” *Journal of Economic Theory*, 31: 251-268.

[25] Gale, D. (1995), “Dynamic Coordination Games” *Economic Theory*, 5: 1-18.

[26] Gale, D. (2001), “Monotone Games with Positive Spillovers” *Games and Economic Behavior*, 37: 295-320.

[27] Gasparini M, R. Clarisó, M. Brambilla, and J. Cabot (2020), “Participation inequality and the 90–9–1 principle in open source” in *Proceedings of the 16th International Symposium on Open Collaboration* 6:1–7. https://doi.org/10.1145/3412569.3412582.

[28] Goeree, J.K. and C.A. Holt (2005), “An explanation of anomalous behavior in models of political participation”, *American Political Science Review*, 99, 201–213.

[29] Haig, J. and C. Cannings (1989), “The n-Person War of Attrition,” *Acta Applicandae Matematicae*, Jannuary/February, 14(1-2): 49-74.

[30] Harrison, G.W., J. Hirschleifer (1989), “An Experimental Evaluation of Weakest Link/Best Shot Models of Public Goods”, *Journal of Political Economy*, 97(1), 201–225.

[31] Heifetz A, R. Heller, and R. Ostreiher (2021), “Do Arabian babblers play mixed strategies in a “volunteer’s dilemma”?”, *Journal of Behavioral and Experimental Economics*, 91: 1-7.

[32] Hellwig, M. (2003), “Public Good Provision with Many Participants”, *The Review of Economic Studies*, 70: 589-614.

[33] Hobsbaum E. and G. Rude (1969), *Capital Swing*, Phoenix Press.

[34] Ledyard, J.O. and T.R. Palfrey (1994), “Voting and Lottery Drafts as Efficient Public Goods Mechanisms,” *The Review of Economic Studies*, 61(2): 327-355.

[35] Lohmann S. (1994), "The dynamics of Informational Cascades: The Monday Demonstrations in Leipzig, East Germany," 1989-1991, World Politics, 47(1): 42-101.

[36] Lockwood, B., and J.P. Thomas (2002): "Gradualism and Irreversibility", *Review of Economic Studies*, 69: 339-356.

[37] Mailath, G.J. and A. Postlewaite (1990), "Asymmetric Information Bargaining Problems with Many Agents." *Review of Economic Studies*, 57:351-67.

[38] Marx, L. and S. Matthews (2000), "Dynamic Voluntary Contribution to a Public Project," *Review of Economic Studies*, 67: 327-358.

[39] Matthews, S. (2013), "Achievable Outcomes of Dynamic Contribution Games". *Theoretical Economics*, 8(2): 365-403.

[40] Myerson, Roger B. (1982), Optimal Coordination Mechanisms in Generalized Principal-Agent Problems. *Journal of Mathematical Economics* 10: 67-81.

[41] Nöldeke G. and J. Peña (2020), "Group size and collective action in a binary contribution game," *Journal of Mathematical Economics*, 88: 42-51.

[42] Otsubo, H. and A. Rapoport (2008), "Dynamic Volunteer's Dilemmas over a Finite Horizon: An Experimental Study". *Journal of Conflict Resolution*, 52(6):961-984.

[43] Palfrey, T.R. and H. Rosenthal (1984), "Participation and the Provision of Public Goods: a Strategic Analysis". *Journal of Public Economics*, 24:171-193.

[44] Palfrey, T.R. and H. Rosenthal (1991), "Testing Game-theoretic Models of Free Riding: New Evidence on Probability Bias and Learning", in *Laboratory Research in Political Economy* (T. Palfrey, Ed.), Ann Arbor: University of Michigan Press, 239–268.

[45] Park, S.A., M. Sestito, E.D. Boorman, et al. (2019), "Neural computations underlying strategic social decision-making in groups", *Nature Communications,* 10, 5287. https://doi.org/10.1038/s41467-019-12937-5

[46] Patel M., B. Raymond, M. Bonsall, S. West (2018), "Crystal toxins and the volunteer's dilemma in bacteria," *Journal of Evolutionary Biology*, 32(4): 310-319, DOI: 10.1111/jeb.13415

[47] Roberto, F., J. Celestino and H. Schulzrinne (2011), "Using a symmetric game based on volunteer's dilemma to improve VANETs multihop broadcast communication." *2011 IEEE 22nd International Symposium on Personal, Indoor and Mobile Radio Communications*: 777-782.

[48] Schneider A. , A.P. Melis and M. Tomasello (2012), "How chimpanzees solve collective action problems", *Proceedings of the Royal Society B*, 279: 4946–4954, doi:10.1098/rspb.2012.1948.

[49] Schelling T. (1960), *The Strategy of Conflict*, Cambridge, Mass: harvard University Press.